VOL. 34, NO. 7, NOV.-DEC., 1961 PRICE 65¢

MATHEMATICS magazine

MM

MATHEMATICS MAGAZINE

ROBERT E. HORTON, Editor

EDITORIAL CORRESPONDENCE should be addressed to the Editor, Robert E. Horton, Department of Mathematics, Los Angeles City College, 855 North Vermont Avenue, Los Angeles 29, California. Articles should be typewritten and double-spaced on 8½ by 11 paper. The greatest possible care should be taken in preparing the manuscript, and authors should keep a complete copy. Figures should be drawn on separate sheets in India ink and of a suitable size for photographing.

NOTICE OF CHANGE OF ADDRESS and other subscription correspondence should be sent to the Executive Director, H. M. Gehman, Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

ADVERTISING CORRESPONDENCE should be addressed to F. R. Olson, Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

The MATHEMATICS MAGAZINE is published by the Mathematical Association of America at Los Angeles, California and Buffalo, New York, bi-monthly except July-August. Ordinary subscriptions are: 1 year $3.00; 2 years $5.75; 3 years $8.50; 4 years $11.00; 5 years $13.00. Members of the Mathematical Association of America may subscribe at the special rate of 2 years for $5.00. Single copies are 65c; reprints, bound 2c per page plus 10c each, provided the order is placed before the article goes to press.

Published with the assistance of the Jacob Houck Memorial Fund of the Mathematical Association of America.

Second class postage paid in Los Angeles, California.

MATHEMATICS MAGAZINE

VOL. 34., NO. 7., NOV.-DEC., 1961

CONTENTS

THE EDITOR'S PAGE

Axiomatics

Mathematicians, probably more than any other group, are acutely aware of the effect that a change in basic assumptions may have on an organized body of thought. Witness the proliferation of new mathematics when mathematicians freed themselves from the restrictions of Euclid's fifth postulate. Much progress in modern mathematics has resulted from a careful re-examination of the basic foundations of the subject. Mathematicians of the nineteenth and twentieth centuries have eagerly attacked these problems with notable results.

Mathematicians who are also teachers of mathematics have two large areas of thought with which to deal. One is the body of mathematics and the other is the educative process. As teachers of mathematics, we are not likely to be successful without a considerable degree of mastery of both areas. How many of us are as careful to be consistent with our beliefs about the nature of man, his purposes, and how he learns as we are with our mathematical assumptions?

The classroom activities consistent with a belief that the mind is a sort of muscle, or a clean page, or a function over a field would be as different as the consequences of no parallels, or one parallel, or many in geometry. Are your teaching practices well founded upon the best that is known about the nature of human learning?

Finally, we are going to be faced with many new influences in the classroom. Teaching machines, educational television, and programmed textbooks are just a few. If we are to judge these things in terms other than emotional, we must know as much as possible about how students learn. Or are you willing to leave such decisions to someone else?

R.E.H.

SOME GROUPS OF LINEAR TRANSFORMATIONS IN THE PLANE

G. H. Lundberg

THE TETRAHEDRAL GROUP

The method used for finding the linear transformations of the tetrahedron which carry the solid into itself was suggested by the discussion on stereographic projection in *Functions of a Complex Variable* by E. J. Townsend (Henry Holt and Co., 1942).

Rotations of the group: The group of the regular tetrahedron consists of twelve movements which carry the tetrahedron into itself. There are four groups of order three and three of order two. Any third order group has an axis of rotation passing through a vertex. This axis is perpendicular to the opposite equilateral triangular face at its centroid. The axis of rotation of a second order group is perpendicular to a pair of opposite edges at their midpoints. A rotation of 120° is of third order, and if a third order rotation of 240° is followed by one of 120°, the result is the identity. A second order group consists of two rotations of 180° each about its axis. Since all possible axes have been considered, rotations of the tetrahedron about them in any order constitute a closed set.

Determination of the invariant points: The regular tetrahedron is inscribed in a sphere of diameter one tangent to the complex plane at the origin, as illustrated in Figure 1.

Points in the complex plane are designated by $x+iy$, while points on the surface of the sphere can be located by use of triple coordinates (x, y, t). Values of x and y are perpendicular distances from the axis of imaginaries and the axis of reals, respectively. Values of t are perpendicular distances above the plane.

The four altitudes of the tetrahedron intersect at a point which is three-fourths of the distance from each vertex to the opposite face. Thus,

$$AO' = \frac{3}{4} AS \ .$$

Hence,

$$AS = \frac{2}{3} \quad \text{and} \quad OS = \frac{1}{3} \, ,$$

the latter of which is the height of the base of the tetrahedron above the plane. The median or altitude CL bisects BD. So, by the Pythagorean theorem,

$$CL = \frac{\sqrt{3}\, CD}{2} \ .$$

Since S is the point of trisection of the medians of the base,

$$SC = \frac{\sqrt{3}}{3} CD \ .$$

The length of one edge of the tetrahedron, found by considering the right triangle ACS, is

$$CD = \frac{\sqrt{6}}{3}.$$

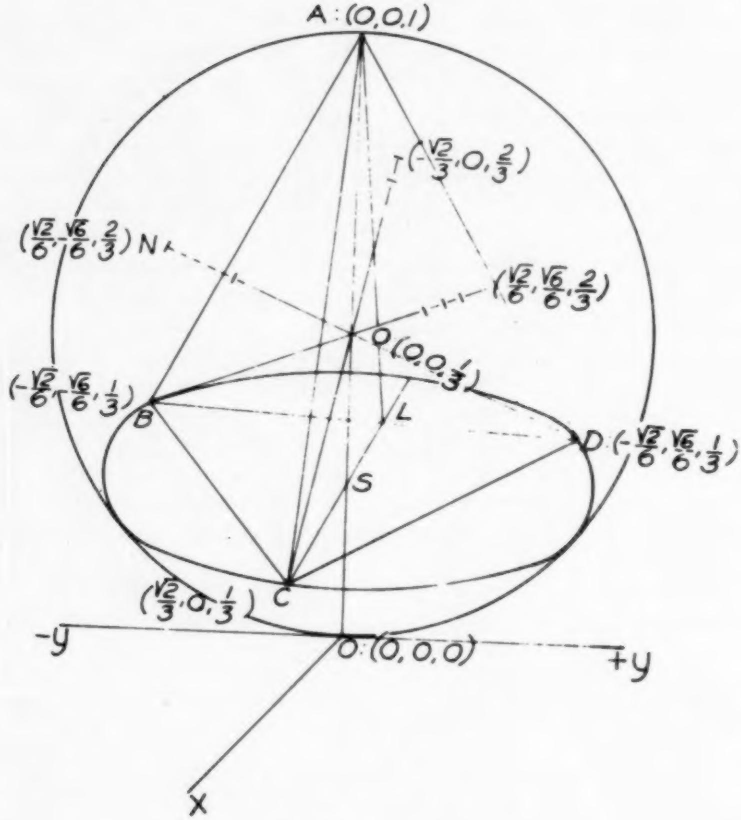


Fig. 1.

Substituting this value of CD, gives

$$CS = \frac{\sqrt{2}}{3} \quad \text{and} \quad CL = \frac{\sqrt{2}}{2}.$$

Since

$$LS = \frac{1}{3}CL\ , \quad LS = \frac{\sqrt{2}}{6}\,.$$

Then, in the right triangle CDL,

$$DL = \frac{\sqrt{6}}{6}\,.$$

From the values of CS, LS, and DL, the coordinates of the vertices B, C, and D are, respectively,

$$\left(\frac{-\sqrt{2}}{6}, \frac{-\sqrt{6}}{6}, \frac{1}{6}\right), \quad \left(\frac{\sqrt{2}}{3}, 0, \frac{1}{3}\right) \quad \text{and} \quad \left(\frac{-\sqrt{2}}{6}, \frac{\sqrt{6}}{6}, \frac{1}{3}\right).$$

When the equation of a line through the center of the sphere is solved simultaneously with the equation of the sphere, the coordinates of the points of intersection on the surface of the sphere are found. To find the equation of the axis through vertex D, the coordinates of D and O' are substituted in the two-point form

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{t-t_1}{t_2-t_1}\,.$$

Then

$$x = \sqrt{2}\,t - \frac{\sqrt{2}}{2} \quad \text{and} \quad y = \sqrt{6}\,t - \frac{\sqrt{6}}{2}\,.$$

Substituting the values of x and y in the equation of the sphere

$$x^2 + y^2 + (t - \tfrac{1}{2})^2 = \tfrac{1}{4}$$

gives

$$9t^2 - 9t + 2 = 0\,,$$

whence

$$t = \frac{2}{3} \quad \text{and} \quad t = \frac{1}{3}\,.$$

When $t = \frac{2}{3}$,

$$x = \frac{\sqrt{2}}{6} \quad \text{and} \quad y = \frac{-\sqrt{6}}{6}\,,$$

and when $t = \frac{1}{3}$,

$$x = \frac{-\sqrt{2}}{6} \quad \text{and} \quad y = \frac{\sqrt{6}}{6}\,.$$

The two sets of values of (x, y, t) are the coordinates of the poles of the axis through D. The coordinates of the poles of other axes can be obtained in a similar manner.

To find the points in the complex plane which correspond to those

on the sphere, it is necessary to use the following formulae which give the relation between the real parts of the complex number, $z = x' + iy'$, in terms of the coordinates of the sphere:

$$x' = \frac{x}{1-t}, \quad y = \frac{x}{1-t}.$$

The axis of rotation through the vertex C has

$$\left(\frac{\sqrt{2}}{3}, 0, \frac{1}{3}\right) \quad \text{and} \quad \left(\frac{-\sqrt{2}}{3}, 0, \frac{2}{3}\right)$$

for the coordinate values of its poles. Substituting the first set of coordinates in the above formulas gives

$$x' = \frac{\sqrt{2}}{2} \quad \text{and} \quad y' = 0,$$

while the second set gives

$$x' = -\sqrt{2} \quad \text{and} \quad y' = 0.$$

The corresponding points on the complex plane of the poles with coordinates

$$\left(\frac{\sqrt{2}}{3}, 0, \frac{1}{3}\right) \quad \text{and} \quad \left(\frac{-\sqrt{2}}{3}, 0, \frac{2}{3}\right)$$

will be

$$z = \frac{\sqrt{2}}{2} \quad \text{and} \quad z = -\sqrt{2}.$$

These points in the complex plane are also the invariant points for any transformation corresponding to any rotation of the sphere with an axis through DC. In a similar manner, the poles with axes through D and B are, respectively,

$$z = \frac{\sqrt{2}\,\omega}{2}, \quad z = -\sqrt{2}\omega \quad \text{and} \quad z = \frac{\sqrt{2}\,\omega^2}{2}, \quad z = -\sqrt{2}\,\omega^2.$$

Transformations of third order groups: The transformation in the complex plane corresponding to rotations of 120°, 240°, and 360° of the sphere with an axis perpendicular to the plane at the origin consists of the dihedral group,

$$z' = z, \quad z' = \omega z, \quad \text{and} \quad z' = \omega^2 z.$$

If the values of z for the invariant points corresponding to the poles of an axis of rotation through vertex C are substituted in

$$cz^2 + (d-a)z - b = 0,$$

the relation of the coefficients is

$$c = b = \sqrt{2}(d-a).$$

Then the general transformation for all rotations of an axis through C becomes

$$z' = \frac{az + \sqrt{2}(d-a)}{\sqrt{2}(d-a)z + d} \, .$$

If D is carried into A the value of z corresponding to D is substituted and the result equated to the value of z, which corresponds to A. Thus,

$$\frac{\sqrt{2}\,\omega + \sqrt{2}\,(d-a)}{\sqrt{2}\,(d-a)\,\frac{\sqrt{2}\omega}{2} + d} = \infty \, .$$

Hence,

$$a = -d\omega$$

and the transformation carrying D into A is

$$z' = \frac{\omega z + \sqrt{2}\,\omega^2}{\sqrt{2}\,\omega^2 z - 1} \, ,$$

which is also the transformation carrying A into B and B into D. In like manner, the transformation which carries D into B, B into A, and A into D, is

$$z' = \frac{\omega^2 z + \sqrt{2}\,\omega}{\sqrt{2}\,\omega z - 1} \, .$$

With the identity $z' = z$, the two transformations make a group of order three.

To bring each transformation to its simplest form requires the use of the relationship which exists among the cube roots of unity

$$\omega^3 + \omega^2 + \omega = 0 \, .$$

Now, to check the above third order group, the first transformation is repeated, giving

$$z' = \frac{\omega\left(\frac{\omega z + \sqrt{2}\,\omega^2}{\sqrt{2}\,\omega^2 z - 1}\right) + \sqrt{2}\,\omega^2}{\sqrt{2}\,\omega^2\left(\frac{\omega z + \sqrt{2}\,\omega^2}{\sqrt{2}\omega^2 z - 1}\right) - 1}$$

which reduces to

$$z'' = \frac{(\omega^2 + 2\omega)z + \sqrt{2}\,(1-\omega^2)}{\sqrt{2}\,(1-\omega^2)z + 2\omega + 1} \, .$$

Dividing each coefficient by $2\omega + 1$ or $-\omega^2 + \omega$ gives

$$z'' = \frac{\omega^2 z + \sqrt{2}\,\omega}{\sqrt{2}\,\omega z - 1} \, .$$

Another substitution brings the identity

$$z''' = z \, .$$

The group of transformations of order three which correspond to rotations about an axis passing through vertex B are similarly found to be

$$z' = \frac{\omega z + \sqrt{2}\,\omega}{\sqrt{2}\,z - 1},$$

which carries A into D, D into C, and C into A;

$$z' = \frac{\omega^2 z + \sqrt{2}}{\sqrt{2}\,\omega^2 z - 1},$$

which carries A into C, D into A, and C into D; and the identity,

$$z' = z .$$

The third order transformations of rotations about an axis passing through vertex D are likewise found to be:

$$z' = \frac{\omega z + \sqrt{2}}{\sqrt{2}\,\omega z - 1},$$

which carries A into C, C into B, and B into A;

$$z' = \frac{\omega^2 z + \omega^2\sqrt{2}}{\sqrt{2}\,z - 1},$$

which carries A into B, C into A, and B into C; and the identity,

$$z' = z .$$

Transformations of second order groups: The remaining three transformations of order two corresponding to rotations of the sphere about axes which pass through a pair of opposite edges may be generated by certain pairs of transformations already found.

The transformation of order two which corresponds to a rotation of the sphere about an axis which passes through the midpoints of edges AD and BC is found by using the transformations

$$z' = \omega z$$

and

$$z' = \frac{\omega^2 z + \sqrt{2}\,\omega}{\sqrt{2}\,\omega z - 1}.$$

The resulting transformation is

$$z' = \frac{z + \sqrt{2}\,\omega}{\sqrt{2}\,\omega^2 z - 1}$$

which carries A into D and B into C. Repeating this transformation gives the identity,

$$z' = z .$$

Similarly, the transformation which corresponds to the rotation of a sphere with an axis through the midpoints of edges AB and CD is found by first using transformation,

$$z' = \omega^2 z ,$$

and then,

$$z' = \frac{\omega z + \omega^2\sqrt{2}}{\sqrt{2}\,\omega^2 z - 1}$$

which gives the transformation,

$$z' = \frac{z + \omega^2\sqrt{2}}{\sqrt{2}\,\omega z - 1}\,.$$

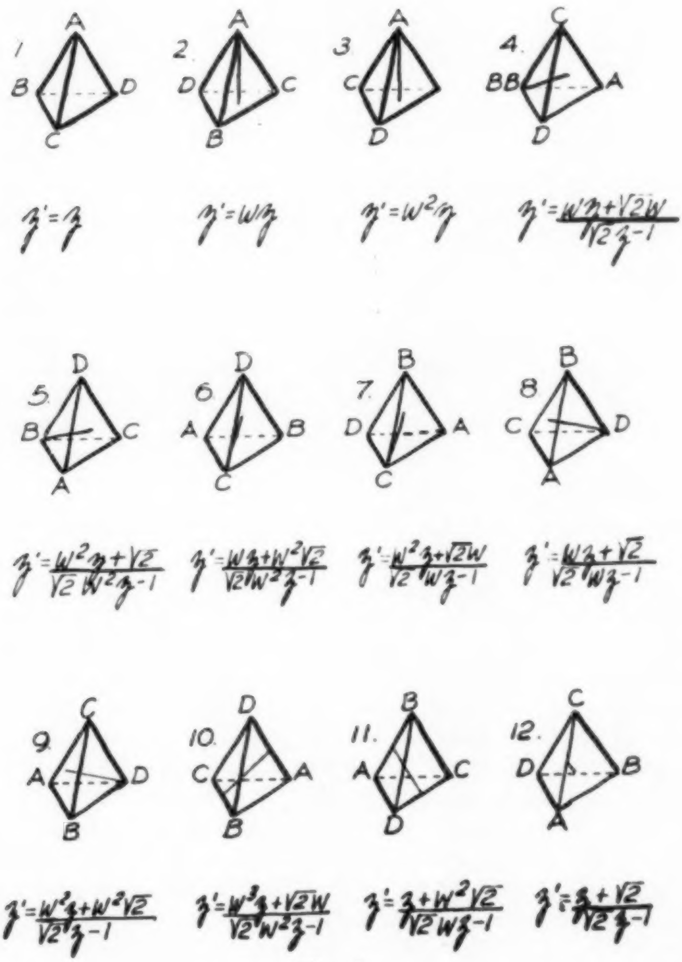


Fig. 2.

It carries A into B and C into D. When this transformation is followed by itself, the result is the identity,

$$z' = z \, .$$

Finally, the third second order transformation, which corresponds to a rotation about an axis passing through the midpoints of AC and BD, is found by using transformations,

$$z' = \omega^2 z$$

and

$$z' = \frac{\omega z + \sqrt{2}}{\sqrt{2}\,\omega z - 1} \, ,$$

which gives

$$z' = \frac{z + \sqrt{2}}{\sqrt{2}\, z - 1} \, .$$

Figure 2 displays the twelve positions which the tetrahedron assumes when carried into itself by the rotations of the sphere and their corresponding transformations in the complex plane.

The entire set of twelve transformations can be generated by transformations 2, 3, 6, and 7, as numbered in the figure. Mention has already been made that 2 followed by 7 gives 10, and that 3 succeeded by 6 produces 11. If the order is 2 and then 6, the result is 9. Nine repeated gives 8. If 8 is preceded by 2, 5 is found. Then 5, followed by itself, gives 4. Finally, 10 and 11 produce 12, which accounts for all transformations of the group. Further, the product of any number of transformations reproduces a transformation of the set.

THE OCTAHEDRAL GROUP

There are twenty-four movements in the group of the regular octahedron which carry the solid into itself. These consist of three groups of order four, four of order three, and six of order two. Each group includes the original position of the octahedron as one member of its set. Any fourth order group has an axis of rotation passing through a pair of opposite vertices. A third order group has its axis of rotation intersecting the opposite faces at their centroids. The axis of a second order group is perpendicular to a pair of opposite edges at their midpoints. Movements of a fourth order group consist of rotations of 90°, 180°, 270°, and 360° about the axis. Third order groups bring the identity when the movements are made up of three rotations of 120° or one of 120° and another of 240°; while two rotations of 180° are sufficient to restore the original position of the octahedron for all second order groups. Since all possible movements carrying the regular octahedron into itself have been considered, the rotations about any of the axes in any order constitute a group. The method of obtaining these transformations is essentially similar to that used for the tetrahedral group.

In Figure 4, the entire group of the twenty-four transformations, together with the corresponding position of the octahedron for each is given. The first twelve transformations can be obtained by the type of analysis

previously used. The remaining twelve may be derived by certain pairs

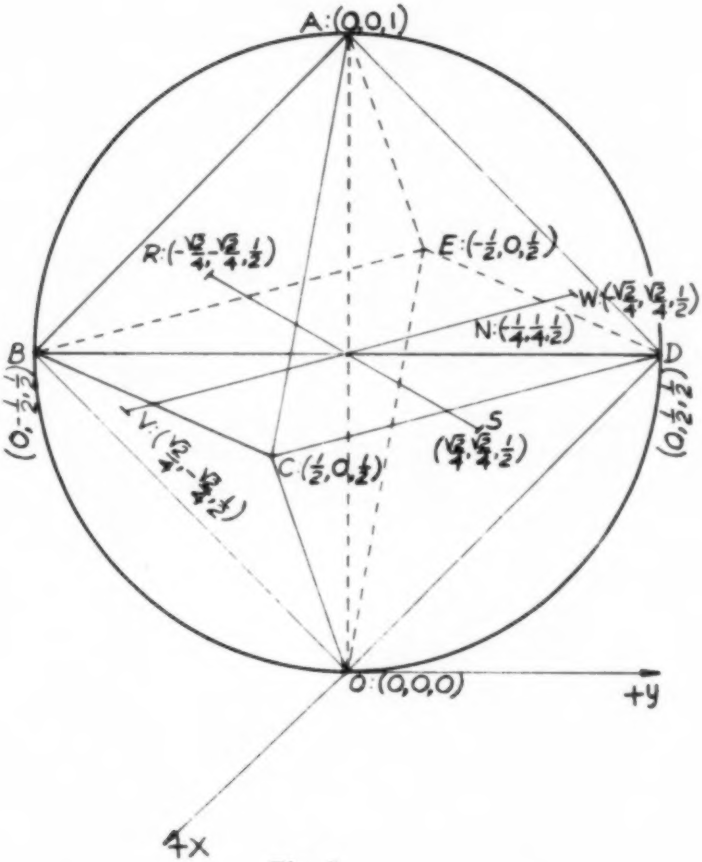


Fig. 3.

of the known transformations, since the distinct rotations about the axes of the octahedron constitute a closed set. Of these, each transformation of numbers 13 to 16, inclusive, with the identity or transformation number one, constitutes a group of order two, while the pairs of transformations numbered 17, 18; 19, 20; 21, 22; and 23, 24, each with the identity, make up the four groups of order three. Now, if transformation number three,

$$z' = -z,$$

is followed by seven,

$$z' = \frac{z+1}{-z+1},$$

then transformation thirteen is found to be

$$z' = \frac{-z+1}{z+1}.$$

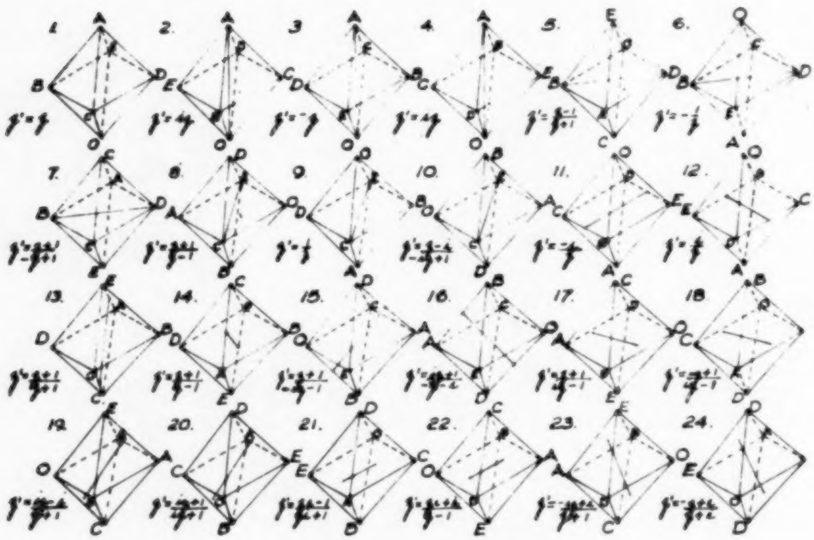


Fig. 4.

In like manner, the second order transformations, 14, 15, 16, may be derived from transformations 9 and then 7, 6 and then 8, and 3 and then 7, respectively.

The third order transformations, 17, 19, 21, and 23, may be found by using the following pairs of transformations: 2, 8; 2, 10; 5, 8; and 13, 2, respectively. The pairs are used in the order given. If each third order transformation 17, 19, 21, and 23 is repeated, the second member of each third order group is found to be 18, 20, 22, and 24. If, again, each of the latter is followed by 17, 19, 21, and 23, respectively, the identity is reached for all.

Since the entire group of rotations can be generated from the two fourth order groups with axes of rotation AD and CE, all the transformations of the octahedral set may be derived from the transformations 2, 3, 4, 8, 9, and 10. Mention has already been made that transformation 8, when used after 3, 2, and 5 gives transformations 16, 17, and 21, respectively. Also, that when 17 and 21 are repeated, their respective results

are 18 and 22. When 4 is used first and then 8, 3 is found; and, as before, if 23 is repeated, the product is 24. Now, 24 followed by 18 will bring 19. One repetition of the latter produces 20. If 23 precedes 2, 5 is obtained. The next member of the fourth order group, 6, is found by repeating 5. Employing 5 after 6 gives 7. When 20 is used before 7, 11 is found. Then, 23 followed by 11, produces 14. Now, if 2 follows 19, then 3 is the result; but if 2 follows 9, the product transformation will be 12. Finally, 21 and then 2 will produce 15 – which accounts for all of the set of twenty-four transformations.

REFERENCE

E. J. Townsend, *Functions of a Complex Variable*, pp. 184-188, Henry Holt and Co., 1942.

Vanderbilt University
Nashville, Tennessee

DETERMINANT WITH SQUARE VALUE INDEPENDENT OF FOUR ELEMENTS

Charles W. Trigg

$$D = \begin{vmatrix} a & b & c & a \\ c & k & k & b \\ b & k & k & c \\ a & c & b & a \end{vmatrix} = (b-c)^2[4ak-(b+c)^2] .$$

If $a = 0$, $D = -(b^2-c^2)^2$, which is independent of k.
If $k = 0$, $D = -(b^2-c^2)^2$, which is independent of a.

Los Angeles City College

CYCLOIDITORY

Wakeling of the rolling wheel,
How you used to weary me
In your trite extensity
With your stale immensity,
With your stopless hipping hopness
And your skipping flipping flopness
Bouncy dropness, bottom topness,
Shorn of all suspensity.

Traceling of the tumbleweed,
I have learned to treasure thee
For the richness found when we
Study you intensively.
I now toast your tautochronic,
Bravo your brachistocronic,
Aye, and skol your isochronic
Pleasurous propensity.

Marlow Sholander

SOME CONGRUENCE PROPERTIES OF THE LEGENDRE POLYNOMIALS*

L. Carlitz

1. Let $P_n(x)$ denote the Legendre polynomial of degree n and let p be an odd prime. The writer [2] has proved the congruences

$$P_p(x) \equiv 2^{-p}\{(x+1)^p + (x-1)^p\} \pmod{p^2}, \tag{1.1}$$

$$P_{2p}(x) - P_p^2(x) \equiv 2^{1-2p}(x^2-1)^p \pmod{p^2}. \tag{1.2}$$

Chatterjea [3] has proved that

$$P_{np} \equiv \sum_{2r\leqq n} c_r (P_{2p} - P_p^2)^r P_p^{n-2r} \pmod{p^2}, \tag{1.3}$$

$$P_p^n \equiv \sum_{2r\leqq n} d_r (P_{2p} - P_p^2)^r P_{(n-2r)p} \pmod{p^2}. \tag{1.4}$$

The coefficients c_r, d_r are integers that satisfy certain linear relations but are not determined explicitly. In view of (1.1) and (1.2), we may rewrite (1.3) and (1.4) in the following form:

$$P_{np} \equiv 2^{-np} \sum_{2r\leqq n} 2^r c_r (x^2-1)^{rp}((x+1)^p + (x-1)^p)^{n-2r} \pmod{p^2}, \tag{1.5}$$

$$2^{-np}((x+1)^p + (x-1)^p)^n \equiv \sum_{2r\leqq n} 2^{r(1-2p)} d_r (x^2-1)^{rp} P_{n-2rp} \pmod{p^2}. \tag{1.6}$$

In the present note we shall show that

$$P_{np} \equiv 2^{-np} \sum_{2r\leqq n} \frac{n!}{r!r!(n-2r)!}(x^2-1)^{rp}((x+1)^p + (x-1)^p)^{n-2r} \pmod{p^2}, \tag{1.7}$$

$$((x+1)^p + (x-1)^p)^n \equiv \sum_{2r\leqq n} 2^{(n-2r)p} a_r \frac{n!}{r!r!(n-2r)!}(x^2-1)^{rp} P_{(n-2r)p} \pmod{p^2}, \tag{1.8}$$

where the a_r are determined by means of

$$\sum_{r=0}^{n} \binom{n}{r}^2 a_r = \begin{cases} 1 & (n=0) \\ 0 & (n>0). \end{cases} \tag{1.9}$$

*Supported in part by National Science Foundation grant G 9425.

2. We shall require the familiar formulas

$$P_n(x) = 2^{-n}\sum_{r=0}^{n}\binom{n}{r}^2(x+1)^r(x-1)^{n-r}, \tag{2.1}$$

$$P_n(x) = 2^{-n}\sum_{2r\leqq n}\frac{n!}{r!r!(n-2r)!}(x^2-1)^r(2x)^{n-2r}. \tag{2.2}$$

Also we shall use the identity [4, p. 177]

$$\alpha^n+\beta^n = \sum_{2r\leqq n}\frac{(-n)_{2r}}{r!(-n+1)_r}(\alpha+\beta)^{n-2r}(\alpha\beta)^r \quad (n\geqq 1), \tag{2.3}$$

where

$$(a)_r = a(a+1)\cdots(a+r-1), \quad (a)_0 = 1.$$

Since

$$\binom{np}{rp}\equiv\binom{n}{p} \pmod{p^2},$$

(2.1) implies

$$\begin{aligned}P_{np} &\equiv 2^{-np}\sum_{r=0}^{n}\binom{n}{r}^2(x+1)^{rp}(x-1)^{(n-r)p}\\ &\equiv 2^{-np}\sum_{2r\leqq n}{}'\binom{n}{r}^2(x^2-1)^{rp}((x+1)^{(r-2r)p}+(x-1)^{(n-2r)p}),\end{aligned}$$

where the prime on the summation sign indicates that for n even the last term is

$$\binom{n}{n/2}^2(x^2-1)^{np/2}.$$

Now using (2.3) we get

$$\begin{aligned}P_{np} &\equiv 2^{-np}\sum_{2r\leqq n}\binom{n}{r}^2(x^2-1)^{rp}\cdot\\ &\qquad\sum_{2s\leqq n-2r}\frac{(-n+2r)_{2s}}{s!(-n+2r+1)_s}(x^2-1)^{sp}((x+1)^p+(x-1)^p)^{n-2r-2s}\\ &\equiv 2^{-np}\sum_{2k\leqq n}(x^2-1)^{kp}((x+1)^p+(x-1)^p)^{n-2k}\cdot\sum_{r+s=k}\binom{n}{r}^2\frac{(-n+2r)_{2s}}{s!(-n+2r+1)_s}.\end{aligned}$$

We put

$$(2.4)\qquad P_{np} \equiv 2^{-np} \sum_{2k \leqq n} A_{n,k}(x^2-1)^{kp}((x+1)^p+(x-1)^p)^{n-2k} \pmod{p^2},$$

where

$$(2.5)\qquad A_{n,k} = \sum_{r+s=k} \binom{n}{r}^2 \frac{(-n+2r)_{2s}}{s!(-n+2r+1)_s}.$$

This sum can be evaluated as the limiting case of a hypergeometric identity [1, p. 28, formula (3)]. However it is simpler to proceed in the following way.

Since

$$(2.6)\qquad P_{np}(x) \equiv P_n(x^p) \pmod{p},$$

(2.2) yields

$$(2.7)\qquad P_{np} \equiv 2^{-n} \sum_{2r \leqq n} \frac{n!}{r!r!(n-2r)!}(x^2-1)^{rp}(2x)^{(n-2r)p} \pmod{p}.$$

Also (2.4) implies

$$(2.8)\qquad P_{np} \equiv 2^{-n} \sum_{2r \leqq n} A_{n,r}(x^2-1)^{rp}(2x)^{(n-2r)p} \pmod{p}.$$

Comparing (2.7) and (2.8) we get

$$A_{n,r} \equiv \frac{n!}{r!r!(n-2r)!} \pmod{p}.$$

This congruence is valid for all primes $p > 2$. Since, by (2.5), $A_{n,r}$ is an integer independent of p we infer that

$$A_{n,r} = \frac{n!}{r!r!(n-2r)!},$$

so that (2.4) reduces to (1.7).

3. The system of linear equations

$$\sum_{2r \leqq n} \frac{n!}{r!r!(n-2r)!} \lambda^r x_{n-2r} = y_n \quad (n = 0, 1, 2, \cdots)$$

evidently has the solution

$$x_n = \sum_{2r \leqq n} a_{n,r} \lambda^r y_{n-2r} \quad (n = 0, 1, 2, \cdots),$$

where λ is an indeterminate and the $a_{n,r}$ are integers. Consequently (1.7) implies

$$(3.1)\qquad ((x+1)^p+(x-1)^p)^n \equiv \sum_{2r\leqq n} a_{n,r}(x^2-1)^{rp}2^{(n-2r)p}P_{(n-2r)p} \pmod{p^2},$$

where of course the $a_{n,r}$ are independent of p. If we replace the modulus by p, and use (2.6), (3.1) becomes

$$(3.2)\qquad (2x)^n \equiv \sum a_{n,r}(x^2-1)^r 2^{n-2r}P_{n-2r} \pmod{p}.$$

On the other hand, it follows from the identity

$$e^{xt}I_0((x^2-1)^{1/2}t) = \sum_{n=0} P_n(x)\frac{t^n}{n!},$$

that

$$(3.3)\qquad x^n = \sum_{2r\leqq n} 2^{-2r}a_r\frac{n!}{r!r!(n-2r)!}(x^2-1)^r P_{n-2r}(x),$$

where

$$(3.4)\qquad \sum_{r=0}^{\infty}\frac{a_r}{2^{2r}}\frac{t^{2r}}{r!r!}I_0(t) = 1.$$

Clearly (3.4) is equivalent to

$$(3.5)\qquad \sum_{r=0}^{n}\binom{n}{r}^2 a_r = \begin{cases}1 & (n=0)\\ 0 & (n>0),\end{cases}$$

from which it is apparent that the a_r are integers. Comparison of (3.3) with (3.2) yields

$$2^{n-2r}a_{n,r} \equiv 2^{n-2r}a_r\frac{n!}{r!r!(n-2r)!} \pmod{p},$$

which implies

$$a_{n,r} = a_r\frac{n!}{r!r!(n-2r)!}.$$

This completes the proof of (1.8).

REFERENCES

1. W. N. Bailey, *Generalized hypergeometric series*, Cambridge, 1935.
2. L. Carlitz, Congruence properties of the polynomials of Hermite, Laguerre and Legendre, Mathematische Zeitschrift, vol. 59 (1954), pp. 474-483.
3. S. K. Chatterjea, On congruence properties of Legendre polynomials, Mathematics Magazine, Vol. 34, No. 6, 1961, pp. 329-336.
4. G. Chrystal, *Algebra*, vol. 2, Edinburgh, 1889.

Duke University

A NOTE ON LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS*

Arthur H. Kruse

1. Introduction. A linear differential equation with constant coefficients can be solved in a routine manner by a number of methods. The purpose of this note is to write explicitly a solution of such an equation in terms of a divided difference (cf. paragraph 2). The divided difference involved will be taken with respect to a parameter; its arguments will be the roots of the characteristic equation of the differential equation. This result is in Theorems 1 and 1′ of paragraph 5. Once formulated, the theorem for the case of distinct roots of the characteristic equation is trivial to verify, but a rigorous proof of the general case, while not at all difficult, relies on non-trivial properties of divided differences, mainly uniform continuity properties. In the interest of clarity and precision this note has been written, insofar as feasible, in the precise language of modern mathematics.

In Theorem 2 of paragraph 5 and its corollary the result of Theorems 1 and 1′ is used to obtain an estimate for that solution of a linear differential equation with constant coefficients satisfying certain initial conditions. Theorem 3 and Remark 3 constitute a novel approach to the (finite) Taylor expansion of a function.

The result of Theorems 1 and 1′ may be derived by essentially Lagrange's method of variation of parameters. This second method of proof leads the writer to suspect that the result is known. However, the writer has been unable to locate it in the literature. Corollary 2 in paragraph 5 is well-known, and most likely Corollary 1 in paragraph 5 is well-known also.

Throughout this note, R is the set of all real numbers, C is the set of all complex numbers, a and b are real numbers with $a < b$, and $[a, b]$ is the closed interval from a to b.

2. Divided differences. For an elementary account of divided differences, cf., e. g., [3, Ch. I]. For a rigorous development of some deeper, but nonetheless fundamental, properties of divided differences, cf. [5].

Given a function $f : [a, b] \to C$ and the non-negative integer n, the *divided difference* of *order* n of f is a function defined on the set of all $(n+1)$-tuples $(x_0, \cdots, x_n)$ such that $a \leqq x_h \leqq b$ for $h = 0, \cdots, n$ and $x_h \neq x_k$ for $h \neq k$. The divided difference of order 0 of f is f itself; its value at $x \in [a, b]$ is written $[x; f] = f(x)$. The divided difference of order n of f with $n > 0$ is defined recursively; its value at $(x_0, \cdots, x_n)$ satisfying the conditions previously stated is

*This paper was presented in part to the Mathematical Association of America on April 21, 1956, and has been supported in part by National Science Foundation Research Grant NSF-G 1126.

$$(1)\qquad [x_0, \cdots, x_n; f] = \frac{[x_0, \cdots, x_{n-1}; f] - [x_1, \cdots, x_n; f]}{x_0 - x_n}\,.$$

The well-known formula

$$(2)\qquad [x_0, \cdots, x_n; f] = \sum_{h=0}^{n} \left\{ \prod_{j=0,\, j\neq h}^{n} (x_h - x_j) \right\}^{-1} f(x_h)$$

is easily proved by induction and shows that divided differences are symmetric in all arguments. If f is n times differentiable with $f^{(n)}$ continuous on $[a, b]$, the divided difference of f of order n is uniformly continuous [5] and may be defined by continuity on the entire set of $(n+1)$-tuples $(x_0, \cdots, x_n)$ with $a \leq x_h \leq b$ for $h = 0, \cdots, n$. Then (1) remains valid for $x_0 \neq x_n$, and

$$(3)\qquad [\underbrace{x, \cdots, x}_{n+1 \text{ arguments}}; f] = \frac{f^{(n)}(x)}{n!} \qquad (a \leq x \leq b)\,.$$

Then $[x_0, \cdots, x_n; f]$ remains symmetric in $x_0, \cdots, x_n$.

The divided difference of functions with complex domain are defined by (1) as in the case for a real domain. Let M be an open subset of C, and let $f : M \to C$ be analytic. Then for each compact set $A \subset M$, $[z_0, \cdots, z_n; f]$ is uniformly continuous for all $z_0, \cdots, z_n \in A$ with $z_h \neq z_k$ for $h \neq k$ (this may be proved from the usual contour integral representation of divided differences [3, p. 11]).

If f is a polynomial function of degree $< n$, the divided difference of order n of f vanishes identically.

3. Divided differences with respect to a parameter. Let M be an open subset of R or C. Given a function (of two variables, the first real and the second real or complex) $\alpha : [a, b] \times M \to C$ such that $\dfrac{\partial^k}{\partial x^k}\alpha(x, m)$ is defined for all $(x, m) \in [a, b] \times M$ (with k a given non-negative integer), for each $x \in [a, b]$ let $\alpha_x : M \to C$ and $D_x^k \alpha_x : M \to C$ be defined by $\alpha_x(m) = \alpha(x, m)$ and $D_x^k \alpha_x(m) = \dfrac{\partial^k}{\partial x^k}\alpha(x, m)$ for each $m \in M$.

It is easy to prove that if $\dfrac{\partial}{\partial m}\alpha(x, m)$ is defined and continuous (in the variable (x, m)) and if $g : [a, b] \to C$ is (Lebesgue) integrable, then

$$\frac{d}{dm}\int_a^b \alpha_x(m) g(x)\,dx = \int_a^b \left\{ \frac{\partial}{\partial m}\alpha_x(m) \right\} g(x)\,dx\,.$$

The following lemma generalizes this. In an application of Lemma 1

(cf. Remark 1 of paragraph 5), m will play the role of a parameter.

LEMMA 1. *Let M be an open subset of R or C. Let the function $\alpha : [a, b] \times M \to C$ be such that $\frac{\partial^j}{\partial m^j} \alpha(x, m)$ is defined and continuous (in the variable (x, m)) for $j = 0, \cdots, n$. Let $g : [a, b] \to R$ be Lebesgue integrable. Let $G : M \to C$ be defined by*

$$G(m) = \int_a^b \alpha_x(m) g(x) dx \qquad (m \in M) .$$

Then

$$[m_0, \cdots, m_n; G] = \int_a^b [m_0, \cdots, m_n; \alpha_x]\, g(x) dx \quad (m_0, \cdots, m_n \in M) .$$

Proof: The proof is by induction on n. If $m_0 = m_1 = \cdots = m_n$, then

$$[m_0, \cdots, m_n; G] = \frac{d^n}{dm^n} G(m) = \int_a^b \frac{d^n}{dm^n} \alpha_x(m) g(x) dx = \int_a^b [m_0, \cdots, m_n; \alpha_x] g(x) dx \ .$$

If not $m_0 = m_1 = \cdots = m_n$, it may be supposed that $m_0 \neq m_n$, and the equation to be proved follows readily from (1) and the induction hypothesis. Q. e. d.

In an application of the following obvious lemma (cf. the proof of Lemma 3 in paragraph 5) m will play the role of a parameter.

LEMMA 2. *Let M be an open subset of R or C. Let the function $\alpha : R \times M \to C$ be such that all partial derivatives of $\alpha(x, m)$ of order $\leqq k$ in $\frac{\partial}{\partial x}$ and of order $\leqq n$ in $\frac{\partial}{\partial m}$ (with k and n given non-negative integers) exist and are continuous (and hence are independent of the order of application of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial m}$) on $R \times M$. Then*

$$[m_0, \cdots, m_n; D_x^k \alpha_x] = \frac{\partial^k}{\partial x^k} \{[m_0, \cdots, m_n; \alpha_x]\} \quad (m_0, \cdots, m_n \in M) .$$

[Moreover, for $m_h \neq m_j$ when $h \neq j$, the equality depends only on the existence of $\frac{\partial^k}{\partial x^k} \alpha(x, m)$.]

Proof: For $k = 0$ or $n = 0$, the equality is trivial. It will be proved for $k = 1$ by induction on n. Suppose (with $k = 1$) that it holds for $n \leqq r$. If $m_0 = m_1 = \cdots = m_{r+1}$, then by (3),

$$\begin{aligned}[m_0, \cdots, m_{r+1}; D_x \alpha_x] &= \frac{\partial^{r+1}}{\partial m^{r+1}} D_x \alpha_x(m)\Big|_{m=m_0} = \frac{\partial^{r+1}}{\partial m^{r+1}} \frac{\partial}{\partial x} \alpha(x, m)\Big|_{m=m_0} \\ &= \frac{\partial}{\partial x} \frac{\partial^{r+1}}{\partial m^{r+1}} \alpha_x(m)\Big|_{m=m_0} = \frac{\partial}{\partial x} [m_0, \cdots, m_{r+1}; \alpha_x] .\end{aligned}$$

By symmetry, if not $m_0 = m_1 = \cdots = m_{r+1}$, it may be supposed that $m_0 \neq m_{r+1}$. If $m_0 \neq m_{r+1}$, then

$$[m_0, \ldots, m_{r+1}; D_x \alpha_x] = \frac{[m_0, \ldots, m_r; D_x \alpha_x] - [m_1, \ldots, m_{r+1}; D_x \alpha_x]}{m_0 - m_{r+1}}$$

$$= \frac{\frac{\partial}{\partial x}[m_0, \ldots, m_r; \alpha_x] - \frac{\partial}{\partial x}[m_1, \ldots, m_{r+1}; \alpha_x]}{m_0 - m_{r+1}}$$

$$= \frac{\partial}{\partial x}[m_0, \ldots, m_{r+1}; \alpha_x] .$$

The induction is complete. The general equality may now be established by induction on k, the inductive step being

$$[m_0, \ldots, m_n; D_x^{k+1} \alpha_x] = [m_0, \ldots, m_n; D_x(D_x^k \alpha_x)] = \frac{\partial}{\partial x}[m_0, \ldots, m_n; D_x^k \alpha_x]$$

$$= \frac{\partial}{\partial x}\frac{\partial^k}{\partial x^k}[m_0, \ldots, m_n; \alpha_x] = \frac{\partial^{k+1}}{\partial x^{k+1}}[m_0, \ldots, m_n; \alpha_x] .$$

Q. e. d.

4. Convolution. Let $f, g : [a, b] \to C$ be continuous functions. The *convolution* $f*g : [a, b] \to C$ of f and g is defined by

$$(4) \qquad f*g(x) = \int_0^{x-a} f(a+u)g(x-u)du \qquad (a \leqq x \leqq b) .$$

The well-known equality $f*g = g*f$ is easily verified by setting $u = x - a - v$ in (4). If g is differentiable on $[a, b]$, then so is $f*g$, and

$$(5) \qquad (f*g)'(x) = f(x)g(a) + f*g'(x) \qquad (a \leqq x \leqq b)$$

by straightforward differentiation of (4). Likewise, if f is differentiable on $[a, b]$, then so is $f*g$, and

$$(6) \qquad (f*g)'(x) = f(a)g(x) + f'*g(x) \qquad (a \leqq x \leqq b) .$$

If f is n times differentiable and g is $n-1$ times differentiable, a simple induction based on (6) yields

$$(7) \qquad (f*g)^{(n)}(x) = \left\{\sum_{j=0}^{n-1} f^{(j)}(a)g^{(n-1-j)}(x)\right\} + f^{(n)}*g(x) \quad (a \leqq x \leqq b)$$

(here $n \geq 1$).

It should be observed that if $f : R \to C$ and $g : [a, b] \to C$ are continuous functions, then the integral $\int_0^{x-a} f(u)g(x-u)du$ (as a function of x) is the convolution of functions $h, g : [a, b] \to C$ with $h(x) = f(x-a)$ for all $x \in [a, b]$. Then (6) becomes

$$(6') \quad \frac{d}{dx}\int_0^{x-a} f(u)g(x-u)du = f(0)g(x) + \int_0^{x-a} f'(u)g(x-u)du \quad (a \leqq x \leqq b)\,.$$

5. The main results. Let $\xi : R \times C \to C$ be defined by $\xi(x, m) = e^{mx}$, and for each $x \in R$ let $\xi_x : C \to C$ be defined by $\xi_x(m) = \xi(x, m) = e^{mx}$ as in paragraph 3. Since ξ_0 is a constant function with value 1,

$$(8) \qquad [m_0, \cdots, m_n; \xi_0] = \begin{cases} 1 & \text{if } n = 0\,, \\ 0 & \text{if } n > 0\,. \end{cases}$$

LEMMA 3. *For all* $m_0, \cdots, m_n \in C$,

$$\left(\frac{\partial}{\partial x} - m_n\right)[m_0, \cdots, m_n; \xi_x] = \begin{cases} 0 & \text{if } n = 0\,, \\ [m_0, \cdots, m_{n-1}; \xi_x] & \text{if } n > 0\,. \end{cases}$$

Proof: The proof for $n = 0$ is trivial. Suppose $n > 0$. If $m_h \neq m_j$ whenever $h \neq j$, then by (2),

$$\begin{aligned}\left(\frac{\partial}{\partial x} - m_n\right)[m_0, \cdots, m_n; \xi_x] &= \sum_{h=0}^{n} \left\{ \prod_{j=0,\, j\neq h}^{n} (m_h - m_j) \right\}^{-1} \left(\frac{\partial}{\partial x} - m_n\right) e^{m_h x} \\ &= \sum_{h=0}^{n} \left\{ \prod_{j=0,\, j\neq h}^{n} (m_h - m_j) \right\}^{-1} (m_h - m_n) e^{m_h x} \\ &= \sum_{h=0}^{n-1} \left\{ \prod_{j=0,\, j\neq h}^{n-1} (m_h - m_j) \right\}^{-1} e^{m_h x} \\ &= [m_0, \cdots, m_{n-1}; \xi_x]\,.\end{aligned}$$

Since

$$\frac{\partial}{\partial x}[m_0, \cdots, m_n; \xi_x] = [m_0, \cdots, m_n; D_x \xi_x]$$

by Lemma 2 and since $D_x\xi_x$ is indefinitely differentiable (for each fixed x), both members of the equation in Lemma 3 are continuous in the argument $(m_0, \cdots, m_n)$ for each fixed $x \in R$. Thus for each fixed $x \in R$ the equation holds for all $(m_0, \cdots, m_n)$, including those with repeated arguments, by continuity. Q. e. d.

LEMMA 4. *Let* $g : [a, b] \to C$ *be continuous, and let* $m_0, \cdots, m_n \in C$. *Then*

$$\left(\frac{d}{dx} - m_n\right)\int_0^{x-a} [m_0, \cdots, m_n; \xi_u] g(x-u)du = \begin{cases} g(x) & \text{if } n = 0\,, \\ \int_0^{x-a} [m_0, \cdots, m_{n-1}; \xi_u] g(x-u)du & \text{if } n > 0\,. \end{cases}$$

Proof: By (6′), (8), and Lemma 3,

$$\left(\frac{d}{dx}-m_n\right)\int_0^{x-a}[m_0, \cdots, m_n; \xi_u]g(x-u)du$$

$$= [m_0, \cdots, m_n; \xi_0]g(x) + \int_0^{x-a}\left\{\left(\frac{d}{du}-m_n\right)[m_0, \cdots, m_n; \xi_u]\right\}g(x-u)du$$

$$= \begin{cases} 1 \cdot g(x) + 0 & \text{if } n = 0 \,, \\ 0 + \int_0^{x-a}[m_0, \cdots, m_{n-1}; \xi_u]g(x-u)du & \text{if } n > 0 \,. \end{cases}$$

Q. e. d.

THEOREM 1. *Let $g : [a, b] \to C$ be continuous, let n be a non-negative integer, and let $m_0, \cdots, m_n \in C$. Then*

$$\left(\frac{d}{dx}-m_0\right)\left(\frac{d}{dx}-m_1\right)\cdots\left(\frac{d}{dx}-m_n\right)\int_0^{x-a}[m_0, \cdots, m_n; \xi_u]g(x-u)du = g(x)$$

for $a \leqq x \leqq b$. Moreover,

$$\frac{d^j}{dx^j}\int_0^{x-a}[m_0, \cdots, m_n; \xi_u]g(x-u)du\,\Big|_{x=a} = 0 \quad (j = 0, \cdots, n) \,.$$

Proof: The first equality follows from Lemma 4 by an easy induction. The proof of the second equality will be by induction also. It is trivial for $j = 0$. Suppose it holds for $j \leqq q < n$. Then by repeated applications of Lemma 4,

$$\frac{d^{q+1}}{dx^{q+1}}\int_0^{x-a}[m_0, \cdots, m_n; \xi_u]g(x-u)du\,\Big|_{x=a}$$

$$= \left(\frac{d}{dx}-m_{n-q}\right)\left(\frac{d}{dx}-m_{n-q+1}\right)\cdots\left(\frac{d}{dx}-m_n\right)\int_0^{x-a}[m_0, \cdots, m_n; \xi_u]g(x-u)du\,\Big|_{x=a}$$

$$= \int_0^{x-a}[m_0, \cdots, m_{n-q-1}; \xi_u]g(x-u)du\,\Big|_{x=a} = 0 \,.$$

Q. e. d.

The following theorem is an alternative statement of Theorem 1.

THEOREM 1′. *Let $g : [a, b] \to C$ be continuous, let n be a non-negative integer, and let $m_0, \cdots, m_n \in C$. Then the convolution*

$$y(x) = \int_0^{x-a}[m_0, \cdots, m_n; \xi_u]g(x-u)du$$

is a solution of the differential equation

$$\left(\frac{d}{dx}-m_0\right)\left(\frac{d}{dx}-m_1\right)\cdots\left(\frac{d}{dx}-m_n\right)y(x)=g(x) \qquad (a \leqq x \leqq b)$$

such that

$$y^{(j)}(a)=0 \qquad (j=0,\cdots,n)\ .$$

REMARK 1. *In Theorems 1 and 1′ let*

$$F_x(m)=\int_0^{x-a} e^{mu}g(x-u)du \qquad (a \leqq x \leqq b,\ m \in C)\ .$$

Then

$$\int_0^{x-a}[m_0,\cdots,m_n;\xi_u]g(x-u)du=[m_0,\cdots,m_n;F_x]$$

for $a \leqq x \leqq b$ by Lemma 1. Thus the divided difference

$$y(x)=[m_0,\cdots,m_n;F_x]$$

is a solution of the differential equation in Theorem 1′.

The following corollary follows from Theorem 1 or Theorem 1′ by taking $m_0=m_1=\cdots=m_n=m$ and applying (3).

COROLLARY 1. *Let $g:[a,b]\to C$ be continuous, let n be a non-negative integer, and let $m \in C$. Then*

$$\left(\frac{d}{dx}-m\right)^{n+1}\int_0^{x-a}\frac{u^n e^{mu}}{n!}g(x-u)du=g(x) \quad (a \leqq x \leqq b)\ .$$

The following well-known corollary follows from Corollary 1 by taking $m=0$.

COROLLARY 2. *Let $g:[a,b]\to C$ be continuous, and let n be a non-negative integer. Then*

$$\frac{d^{n+1}}{dx^{n+1}}\int_0^{x-a}\left(\frac{u^n}{n!}\right)g(x-u)du=g(x) \quad (a \leqq x \leqq b)\ .$$

THEOREM 2. *Let $g:[a,b]\to C$ be continuous, let n be a non-negative integer, and let $m_0,\cdots,m_n \in C$. Let $y:[a,b]\to C$ be that solution of the differential equation*

$$\left(\frac{d}{dx}-m_0\right)\left(\frac{d}{dx}-m_1\right)\cdots\left(\frac{d}{dx}-m_n\right)y(x)=g(x) \quad (a \leqq x \leqq b)$$

such that

$$y^{(j)}(a)=0 \qquad (j=0,\cdots,n)\ .$$

Then, with r the maximum of the real parts of $m_0,\cdots,m_n$,

$$|y(x)| \leqq \left(\frac{1}{n!}\right)\int_0^{x-a} u^n e^{ru}|g(x-u)|du \quad (a \leqq x \leqq b)\ .$$

(Also, cf. Remark 2 below.)

Proof: The function y is defined as in Theorem 1′. By a known result [4, pp. 8-12], for each $u \in [a, b]$ there are s, $m \in C$ such that $|s| \leq 1$, m is in the convex set generated by $m_0, \cdots, m_n$, and

$$[m_0, \cdots, m_n; \xi_u] = \frac{|s| \xi_u^{(n)}(m)}{n!} = \frac{|s| u^n e^{mu}}{n!} \leqq \frac{u^n e^{ru}}{n!} .$$

The theorem follows. Q. e. d.

In the following corollary f plays the role of an approximation to a solution of a differential equation. The corollary gives an estimate for the error of the approximation. The proof of the corollary is left to the reader.

COROLLARY 3. *Let $g : [a, b] \to C$ be continuous, let n be a positive integer, let $a_0, \cdots, a_n \in C$, and let $y : [a, b] \to C$ be a solution of the differential equation*

$$\sum_{j=0}^{n} a_j y^{(j)}(x) = g(x) \qquad (a \leq x \leq b) .$$

Let $f. : [a, b] \to C$ be such that $f^{(n)}$ is defined and continuous on $[a, b]$ and such that

$$f^{(j)}(a) = y^{(j)}(a) \qquad (j = 0, \cdots, n-1) .$$

Let r be the maximum of the real parts of the zeros of the characteristic polynomial P defined by

$$P(m) = \sum_{j=0}^{n} a_j m^j \qquad (m \in C) .$$

Then

$$|f(x) - y(x)| \leqq \left(\frac{1}{(n-1)!}\right) \int_0^{x-a} u^{n-1} e^{ru} |g(x-u) - \sum_{j=0}^{n} a_j f^{(j)}(x-u)| du$$

for each $x \in [a, b]$. (Also, cf. Remark 2 below.)

REMARK 2. *Let c and r be positive real numbers. Then*

$$\int_0^c u^n e^{ru} du < \frac{e^{rc} c^{n+1}}{n+1} ,$$

$$\int_0^c u^n e^{-ru} du < \frac{e^{-rc} c^{n+1}}{n+1} + \frac{r e^{rc} c^{n+2}}{(n+1)(n+2)} = \frac{e^{-rc} c^{n+1}}{n+1} \left(1 + \frac{r e^{2rc} c}{n+2}\right) .$$

[Proof: Differentiate each member with respect to c.] Thus in Theorem 2,

$$|y(x)| \leqq \begin{cases} \dfrac{e^{r(x-a)}(x-a)^{n+1}}{(n+1)!} \sup\limits_{a \leqq u \leqq x} |g(u)| & \text{if } r \geqq 0\,, \\[2ex] \dfrac{e^{r(x-a)}(x-a)^{n+1}}{(n+1)!}\left(1 + \dfrac{-re^{-2r(x-a)}(x-a)}{n+2}\right) \sup\limits_{a \leqq u \leqq x} |g(u)| & \text{if } r < 0\,. \end{cases}$$

A similar inequality holds in connection with Corollary 3.

THEOREM 3. *Let $f : [a, b] \to C$ have a continuous nth derivative, and let*

$$R_{n+1}(x) = f(x) - \sum_{j=0}^{n} \left(\frac{f^{(j)}(a)}{j!}\right) x^j \qquad (a \leqq x \leqq b)\,.$$

(Thus R_{n+1} is the remainder in a finite Taylor expansion of f.) Let $a_0, \cdots, a_{n+1} \in C$, and let $m_0, \cdots, m_n$ be the zeros (multiplicities counted) of the polynomial $\sum_{j=0}^{n+1} a_j m^j$ in m. Then

$$R_{n+1}(x)$$
$$= \int_0^{x-a} [m_0, \cdots, m_n; \xi_u] \left\{ \sum_{j=0}^{n+1} a_j f^{(j)}(x-u) - \sum_{k=0}^{n} \frac{1}{k!} \left(\sum_{j=k}^{n} a_{j-k} f^{(j)}(a) \right) (x-u)^k \right\} du$$

for all $x \in [a, b]$.

Proof: Routine calculation shows that $y = R_{n+1}$ is that solution of the differential equation in Theorem 1′ with

$$g(x) = \sum_{j=0}^{n+1} a_j f^{(j)}(x) - \sum_{k=0}^{n} \frac{1}{k!} \left(\sum_{j=k}^{n} a_{j-k} f^{(j)}(a) \right) x^k$$

such that $y^{(j)}(a) = 0$ for $j = 0, \cdots, n$. Thus Theorem 3 follows from Theorem 1′. Q. e. d.

REMARK 3. *In Theorem 3 let $a_{n+1} = 1$ and $a_j = 0$ for $0 \leqq j \leqq n$. Then $m_j = 0$ for $0 \leqq j \leqq n$, and, by (3),*

$$R_{n+1}(x) = \int_0^{x-a} [m_0, \cdots, m_n; \xi_u] f^{(n+1)}(x-u)\,du = \int_0^{x-a} \left(\frac{u^n}{n!}\right) f^{(n+1)}(x-u)\,du$$

for all $x \in [a, b]$. This is the usual formula for R_{n+1}.

6. Variation of parameters. If $y_1, \cdots, y_n$ are $n-1$ times differentiable complex-valued functions on $[a, b]$, the *Wronskian* $W(y_1, \cdots, y_n) : [a, b] \to C$ of $y_1, \cdots, y_n$ is defined by

$$W(y_1, y_2, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ & & \cdots & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} .$$

The following proposition embodies the general result obtainable by Lagrange's method of variation of parameters for solving linear differential equations. Equivalent statements of it and proofs may be found, e.g., in [1, pp. 88-87] and [2, p. 76]. It is the basis of alternative proofs (presented in the next section) of Theorems 1 and 1'.

PROPOSITION. *Let*

$$y^{(n)}(x) + P_1(x)y^{(n-1)}(x) + P_2(x)y^{(n-2)}(x) + \cdots + P_n(x)y(x) = Q(x)$$

be a differential equation with continuous (complex) coefficients $P_1, \cdots, P_n, Q$ *on* $[a, b]$. *Let* $y = y_1,\ y = y_2, \cdots, y = y_n$ *be solutions of*

$$y^{(n)}(x) + P_1(x)y^{(n-1)}(x) + P_2(x)y^{(n-2)}(x) + \cdots + P_n(x)y(x) = 0$$

on $[a, b]$ *such that* $W(y_1, \cdots, y_n)(x) \neq 0$ *for each* $x \in [a, b]$. *Then*

$$y(x) = (-1)^n \sum_{j=1}^{n} (-1)^j y_j(x) \int_a^x \frac{W(y_1, \cdots, y_{j-1}, y_{j+1}, \cdots, y_n)(u)}{W(y_1, \cdots, y_n)(u)} Q(u)\,du$$

defines that solution of the former equation on $[a, b]$ *such that*

$$y^{(j)}(a) = 0 \qquad (j = 0, \cdots, n-1) .$$

The following lemma follows from the well-known formula for Vandermonde determinants.

LEMMA 5. *For* $j = 1, \cdots, n$, *let* $m_j \in C$, *and let* $y_j(x) = e^{m_j x}$ *for each* $x \in [a, b]$. *Then*

$$W(y_1, y_2, \cdots, y_n) = \{ \prod_{h<k} (m_k - m_h) \} y_1 y_2 \cdots y_n .$$

7. Alternative proof of Theorems 1 and 1'. Suppose $g : [a, b] \to C$ is a continuous function, and suppose $m_0, \cdots, m_n \in C$ with $m_h \neq m_k$ whenever $h \neq k$. Let $y_j(x) = e^{m_j x}$ for all $x \in [a, b]$ for $j = 0, \cdots, n$. Then $y = y_0,\ y = y_1, \cdots, y = y_n$ are solutions of the differential equation

$$(\frac{d}{dx} - m_0)(\frac{d}{dx} - m_1) \cdots (\frac{d}{dx} - m_n) y(x) = 0 ,$$

and $W(y_0, y_1, \cdots, y_n)(x) \neq 0$ for each $x \in [a, b]$ by Lemma 5. Thus, by the proposition in paragraph 6 and Lemma 5, that solution of the differential equation

$$(\frac{d}{dx} - m_0)(\frac{d}{dx} - m_1) \cdots (\frac{d}{dx} - m_n) y(x) = g(x) \qquad (a \leqq x \leqq b)$$

Continued on page 409.

FINITE DIFFERENCES AND COMPUTATION OF POLYNOMIALS[1]

Luis de Greiff B.

1. *Introduction.* We shall consider sequences (u_i) or

$$u_1, u_2, \cdots, u_n, \cdots \tag{1.1}$$

of real and complex numbers and we shall calculate with them formally as with infinite dimensional vectors. Thus $k(u_i)$ will be the sequence whose terms are ku_i, i. e. $k(u_i) = (ku_i)$ and similarly $(u_i)+(v_i) = (u_i+v_i)$.

Let us recall that the pth ($p = 1, 2, \cdots$) differences $\Delta_p u_i$ of (1.1) are defined recursively by

$$\Delta_1 u_1 = u_2 - u_1\,, \quad \Delta_1 u_2 = u_3 - u_2\,, \quad \cdots, \quad \Delta_1 u_i = u_{i+1} - u_i\,, \quad \cdots$$

and

$$\Delta_p u_1 = \Delta_{p-1} u_2 - \Delta_{p-1} u_1\,, \quad \Delta_p u_2 = \Delta_{p-1} u_3 - \Delta_{p-1} u_2\,, \quad \cdots\,,$$
$$\Delta_p u_i = \Delta_{p-1} u_{i+1} - \Delta_{p-1} u_i\,, \cdots$$

We shall consider Δ_p as an operator operating on the sequences (u_i) and write $\Delta_p(u_i)$ for the sequence $(\Delta_p u_i)$. It can be seen by induction on p that

$$\Delta_p k(u_i) = k\Delta_p(u_i) \tag{1.2}$$

and

$$\Delta_p\{(u_i)+(v_i)\} = \Delta_p(u_i) + \Delta_p(v_i)\,. \tag{1.3}$$

2. *Expression of the differences by means of the terms u_i.* Finite differences of any order can be expressed by means of the generating sequence. For $p = 2$ we have

$$\Delta_2 u_i = \Delta_1 u_{i+1} - \Delta_1 u_i = u_{i+2} - u_{i+1} - u_{i+1} + u_i = u_{i+2} - 2u_{i+1} + u_i\,,$$

which can be symbolically written as

$$\Delta_2 u_i = (u-1)^{(2)} u_i\,.$$

Here the right hand side is obtained from the ordinary product $(u-1)^2 u_i$ by changing the terms $u^r u_i$ into u_{r+i}. In the general case we have the formula

$$\Delta_p u_i = (u-1)^{(p)} u_i\,, \tag{2.1}$$

or, symbolically,

$$\Delta_p(u_i) = (u-1)^{(p)}(u_i)\,. \tag{2.2}$$

Let us assume that (2.1) is true. Then

$$\Delta_{p+1}u_i = \Delta_p u_{i+1} - \Delta_p u_i = (u-1)^{(p)}u_{i+1} - (u-1)^{(p)}u_i = (u-1)^{(p)}(u_{i+1} - u_i)$$
$$= (u-1)^{(p)}(u-1)^{(1)}u_i = (u-1)^{(p+1)}u_i \ .$$

Since the formula (2.1) holds for $p = 1$ and $p = 2$, we have established its validity for every p.

3. *Sequences of powers of an arithmetical progression.* Let

$$q_0 = a \ , \quad q_1 = a+r \ , \quad q_2 = a+2r \ , \quad \cdots \ , \quad q_i = a+ir \ , \quad \cdots$$

be an arithmetical progression and consider the sequence

(3.1) $$q_0^s, \ q_1^s, \ \cdots, \ q_i^s, \ \cdots,$$

where s is a positive integer. We shall prove the following

THEOREM. The sequence of the $(s+1)$ th differences of the sequence (3.1) is zero, i. e.

$$\Delta_{s+1}(q_i^s) = 0 \ .$$

It follows immediately from the theorem that

(3.2) $$\Delta_{s+k}(q_i^s) = 0$$

for all integers $k \geqq 1$.

Proof. The theorem is true for $s = 1$, since then $\Delta_1 q_i = r$ and $\Delta_2 q_i = 0$. Let us suppose that it is true for s. Then we have

$$\Delta_{s+2}q_i^{s+1} = \Delta_{s+1}(q_i + r)^{s+1} - \Delta_{s+1}q_i^{s+1}$$
$$= \Delta_{s+1}\left(q_i^{s+1} + \binom{s+1}{1}q_i^s r + \binom{s+2}{2}q_i^{s-1}r^2 + \cdots + r^{s+1}\right) - \Delta_{s+1}q_i^{s+1}$$
$$= \binom{s+1}{1} r\Delta_{s+1}q_i^s + \binom{s+2}{2} r^2\Delta_{s+1}q_i^{s-1} + \cdots + r^{s+1}\Delta_{s+1}1 = 0$$

by the hypothesis of induction and the remark following the theorem. The proof is complete.

4. *Expression of the terms u_i by means of the differences.* According to the definition of finite differences we have

$$u_i - u_{i-1} = \Delta_1 u_{i-1} \ ,$$

whence

$$u_i = u_{i-1} + \Delta_1 u_{i-1} \ ,$$

which we shall write formally as

$$u_i = (1 + \Delta_1)^{(1)}u_{i-1} \ .$$

By recursion we obtain

$$u_i = (1 + \Delta_1)^{(i-1)}u_1$$

and more generally

$$u_{k+i} = (1+\Delta_1)^{(i)} u_k \,.$$

5. *Summation of the terms of a sequence.* In order to compute the finite sum

$$S_n = \sum_{i=1}^{n} u_i$$

we write

$$S_n = \sum_{i=1}^{n} u_i = \sum_{i=1}^{n} (1+\Delta_1)^{(i-1)} u_1 = \left\{ \sum_{i=1}^{n} (1+\Delta_1)^{(i-1)} \right\} u_1 = \frac{(1+\Delta_1)^{(n)}-1}{(1+\Delta_1)^{(1)}-1} u_1$$

and we obtain

$$S_n = \frac{(1+\Delta_1)^{(n)}-1}{\Delta_1} u_1 \,. \tag{5.1}$$

Performing the operations indicated in (5.1) we have

$$S_n = \binom{n}{1} u_1 + \binom{n}{2} \Delta_1 u_1 + \binom{n}{3} \Delta_2 u_1 + \cdots + \binom{n}{n} \Delta_{n-1} u_1 \,.$$

6. *The computation of polynomials.* Let us now apply the above theory to a polynomial

$$f(x) = a_0 x^s + a_1 x^{s-1} + \cdots + a_{s-1} x + a_s$$

with real or complex coefficients.

Suppose that the variable x has been given values in an arithmetical progression

$$x_0,\ x_0+h,\ x_0+2h,\ \cdots,\ x_0+ih,\ \cdots$$

Setting $f(x_0+ih) = f_i$ we obtain the sequence of corresponding values

$$f_0,\ f_1,\ f_2,\ \cdots,\ f_i,\ \cdots$$

It follows immediately from (1.2), (1.3) and (3.2) that

$$\Delta_{s+1}(f_i) = 0 \,.$$

By virtue of formula (2.2) this gives us

$$(f-1)^{(s+1)}(f_i) = 0$$

and in particular for $i = 0$

$$f_{s+1} - \binom{s+1}{1} f_s + \binom{s+1}{2} f_{s-1} - \binom{s+1}{3} f_{s-2} + \cdots + (-1)^{s+1} f_0 = 0$$

or, finally

(6.1) $$f_{s+1} = \sum_{\nu=1}^{s+1} (-1)^{\nu-1} \binom{s+1}{\nu} f_{s+1-\nu} .$$

The right hand side can be considered as the scalar product of the vector with components $(-1)^{\nu-1}\binom{s+1}{\nu}$ and of the vector with components $f_{s+1-\nu}$.

This formula makes it possible to automatize the computation of polynomials.[2]

In the case of a cubic polynomial ($s = 3$) the formula (6.1) becomes

(6.2) $$f_4 = 4f_3 - 6f_2 + 4f_1 - f_0 .$$

Let us apply this formula to the computation of the polynomial

$$f(x) = x^3 - 16x^2 + 55x - 24 .$$

A direct calculation yields

$$f(0) = -24 , \quad f(1) = 16 , \quad f(2) = 30 , \quad f(3) = 24 .$$

Substituting these values in (6.2) we have

$$f(4) = 4 \times 24 - 6 \times 30 + 4 \times 16 + 24 = 4$$

and the computation can proceed indefinitely.

As our next example consider the polynomial

$$f(x) = x^6 - 5x^5 + 8x^4 - 12x^3 - 6x^2 + 9x - 7 .$$

Direct computation gives the seven initial values

$$f(-1) = 4 , \quad f(0) = -7 , \quad f(1) = -12 , \quad f(2) = -77 , \quad \text{etc.}$$

These values figure in the third column of Table I below. In the first column we have written the binomial coefficients $\binom{7}{\nu}$ with alternating signs. The computation consists in multiplying scalarly the first column with seven consecutive values of $f(x)$.

Table I

1	$f(-1) =$	4	4	−7	−12	−77	−196
−7	$f(0) =$	−7	49	84	539	1372	−1323
21	$f(1) =$	−12	−252	−1617	−4116	3969	71148
−35	$f(2) =$	−77	2695	6860	−6615	−118580	−538405
35	$f(3) =$	−196	−6860	6615	118580	538405	1696380
−21	$f(4) =$	189	−3969	−71148	−323043	−1017828	−2616789
7	$f(5) =$	3388	23716	107681	339276	872263	1956668
			15383	48468	124609	279524	567483
			$f(6)$	$= f(7)$	$= f(8)$	$= f(9)$	$= f(10)$

In the first row we have the values of $f(x)$ previously obtained; in the second row we have the values of $f(x)$, except the first, multiplied by -7; in the third row we have the values of $f(x)$, except the first and the second, multiplied by 21, and so on.

The computation of the same polynomial for decimal values of x appears in Table II.

Table II

1	$f(2.0) = -77.000000$	−77.000000	−88.546129	−100.912896	−113.952461
−7	$f(2.1) = -88.546129$	619.822903	706.390272	797.667227	892.187968
21	$f(2.2) = -100.912896$	−2119.170816	−2393.001681	−2676.563904	−2963.953125
−35	$f(2.3) = -113.952461$	3988.336135	4460.939840	4939.921875	5412.547840
35	$f(2.4) = -127.455424$	−4460.939840	−4939.921875	−5412.547840	−5862.782135*
−21	$f(2.5) = -141.140625$	2963.953125	3247.528704	3517.669281*	3762.513216
7	$f(2.6) = -154.644224$	−1082.509568	−1172.556427*	−1254.171072	−1322.561303
		−167.508061 $= f(2.7)$	−179.167296 $= f(2.8)$	−188.937329 $= f(2.9)$	−196.000000 $= f(3.0)$

In the actual computation, as soon as a new value for $f(x)$ has been found, we multiply it successively by the binomial coefficients and enter the products in the table diagonally to the right of the entries already written down. Since every binomial coefficient (except 1) figures twice, the number of multiplications to be performed is reduced by one half. In Table II we indicated the multiples of $f(2.7)$ by asterisks.

In Table II we computed the entries exactly. In Table III the same computation is presented with only four decimal places worked out. A very slight systematic divergence ($1/10^5$ approx.) can be observed.

Table III

1	−77.0000	−77.0000	−88.5461	−100.9129	−113.9525
−7	−88.5461	619.8227	706.3903	797.6675	892.1878
21	−100.9129	−2119.1709	−2393.0025	−2676.5634	−2963.9526
−35	−113.9525	3988.3375	4460.9390	4939.9210	5412.5470
35	−127.4554	−4460.9390	−4939.9210	−5412.5470	−5852.7275
−21	−141.1406	2963.9526	3247.5282	3517.6365	3762.3096
7	−154.6442	−1082.5094	−1172.5455	−1254.1032	−1322.3105
		−167.5065	−179.1576	−188.9015	−195.8987

FOOTNOTES

1. This is an abbreviated English version of the author's paper: *Aplicación de la teoria de diferencias finitas al cálculo de polinomios,* Revista de la Academia Colombiana de Ciencias, vol. 9 (1956) pp. 269-273.

2. The author has found the same result earlier following a different procedure, see the booklet: *Deducción de una fórmula para la interpolación y sus aplicaciones al álgebra*, Medellín, Colombia, 1951.

Facultad Nacional de Minas, Medellin, Colombia

PERMUTATIONS FROM 1961

Charles W. Trigg

1) There are twelve distinct permutations of the digits of 1961. Three of these, 1619, 6911, and 9161 are prime. The other nine are non-square composite numbers, only one of which, 9116, has a factor in common with 1961.

2) Twelve distinct three-digit integers can be formed from the digits of 1961. Three of these, 169, 196, and 961, are squares, and their sum is $2 \cdot 3 \cdot 13 \cdot 17$. Five of the integers, 116, 119, 161, 611, and 916, are otherwise composite and their sum 1823 is prime. The other four, 191, 619, 691, and 911, are prime and their sum is $2^2 \cdot 3^2 \cdot 67$.

3) Seven distinct two-digit integers can be formed from the digits of 1961. One of these, 16, is a square. Three, 96, 69, and 91, are otherwise composite and their sum is 16^2. The other three, 11, 19 and 61 are prime and $11 + 19 + 61 = 91$.

4) $1961 - 1691 = 270$, $270 - 072 = 198$, $891 - 198 = 693$,
$693 - 396 = 297$, $792 - 297 = 495$, and $594 - 495 = 99$,
a palindromic number after six subtractions.

5) $1961 + 1691 = 3652$, $3652 + 2563 = 6215$,
$6215 + 5126 = 11341$, and $11341 + 14311 = 25652$,
a palindromic number after four additions.

Los Angeles City College

TEACHING OF MATHEMATICS

Edited by

Rothwell Stephens

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to *Rothwell Stephens Mathematics Department, Knox College, Galesburg, Illinois.*

A MATHEMATICAL PROOF – WHAT IT IS AND WHAT IT SHOULD BE

Robert Alan Melter

Most of the effort expended in Mathematics is devoted to inventing valid proofs for hitherto unproved propositions. In order that this effort be well spent it is reasonable to ask what a mathematical proof is, and even more, what it should be.

There are two basic notions in the air about just what constitutes a mathematical proof. On the one hand are the Formalists who say that a proof consists of a set of statements, the last of which is the statement or proposition to be verified; they say that an arbitrary scheme, e. g., an electronic computer, can be set up to determine whether this set of statements was prepared according to a set of rules known as the propositional and predicate calculus. If the proof was so prepared a green light shines on the computer console, and we are entitled to elevate the proposition to the rank of theorem, and the set of statements becomes a valid mathematical proof. The opposing party also believes that a proof is a series of statements, but, for them, the validity of the proof depends on the opinion of a group of human beings known as professional mathematicians. Thus the correctness of a proof becomes a subjective judgement. We find that, like critics in other fields, mathematicians generally concur in these judgements. Some music critics may feel that Bach is more subtle than Beethoven, but all would agree that they are both composers of the first rank. Similarly, some mathematical proofs are considered to be more awkward than others, but there is seldom continuing disagreement about the ultimate validity of a proof.

We shall see that both of these points of view are too narrow, and that a third, incorporating parts of each, is needed if we are to understand what a mathematical proof should be. In spite of the layman's term, ''electronic brain'', electronic computers do not think. Every operation performed by them must be carefully planned in greatest detail by the computer programmer. Fortunately human beings are equipped with minds which enable them to consolidate some of the steps which a computer must perform discretely. Thus it is unreasonable to ask a mathematician to write out his proofs in the elaborate detail of the Principia Mathematica. But he should realize that if he could not convince the mathematical world by means of this less-detailed kind of proof, he could, as a last resort,

prepare his statements for the "proof-checking" machine, and convince his dubious colleagues beyond any peradventure of a doubt. But this is not enough, for proof checking machines can not evaluate the elegance of proofs, nor can they differentiate between valid proofs for the same theorem. Thus, if we depended upon formalist methods exclusively, after one valid proof had been given for a theorem, the theorem would lose all heuristic value. (At least with respect to devising "better" proofs.) Happily Professor Church has demonstrated that the invention of mathematical proofs can not, in general, be the result of a purely mechanical procedure, but is requisite of human ingenuity; this is a consequence of the fact that no decision procedure exists for the lower functional calculus. However, if subjective, or even artistic human judgements are not made about the quality of mathematical proofs, then the Deduction Theorem would imply that Mathematics is but a difficult jigsaw puzzle.

Neither must mathematical proofs be judged only by a method which depends on human insights alone. The reason for this is both practical and philosophical. Practically, certain mathematical results are closely related to nature. Even the most elegant attempt to prove the general negation of the Theorem of Pythagoras, as it relates to Euclidean Space, must be discarded because we know that Euclidean Geometry corresponds to our (non-relativistic) observations of space and measurement in nature. Since the Theorem of Pythagoras can be verified, to any desired degree of accuracy, by physical measurement, any proof of the "proposition" that, "in any right triangle the sum of the squares of the legs is unequal to the square on the hypotenuse," would imply that it is impossible to make accurate measurements, and therefore to learn about the physical world by observation. Another practical difficulty would be the disappearance of the class of persons known as Ph. D. candidates in Mathematics. Who, indeed, would venture to write a thesis knowing that his efforts were to be judged in the same manner as a pie at a County Fair. Philosophically, Mathematics should be a field in which results, once verified and accepted, will not be discarded at a later date simply because the manner in which these results were obtained is no longer in vogue. Styles change in any part of life in which subjective judgments are made, and so too in Mathematics. Mathematics has become the extensive body of knowledge that it is because Von Neumann could look back upon the proofs of Apollonius and though he may have found them clumsy, he nevertheless accepted their validity.

What then should a mathematical proof be? What it should be includes a fortiori what it is. A mathematical proof *is* a set of statements which can be reformulated to bring on a green light in a "proof-checking" machine. It *should be* possessed of something more — elegance. Just as a music critic may describe a particular composition as delicious, so the mathematical referee may say that a proof is truly delightful, or exciting. This is the part of mathematics that defies quantification; but indeed it adds the greatest joy to mathematical study and research. Elegance can

neither be explained nor taught, except by osmosis. Perhaps the best idea of what a mathematical proof should be can be gotten from an analogy with Architecture. If an architect designs a house with four walls and a roof it will serve its purpose. If he incorporates into it new notions of construction and design it will stimulate the growth of the science and art of Architecture. Thus mathematical proofs should be heuristic and elegant in addition to being merely verifyable.

University of Missouri

LINEAR DIFFERENTIAL EQUATIONS *(Continued from page 400.)*

for which $y^{(j)}(a) = 0$ for $j = 0, \cdots, n$ is given by

$$y(x) = (-1)^n \sum_{j=0}^{n} (-1)^j y_j(x) \int_a^x \frac{W(y_0, \cdots, y_{j-1}, y_{j+1}, \cdots, y_n)(u)}{W(y_0, \cdots, y_n)(u)} g(u)\, du$$

$$= (-1)^n \sum_{j=0}^{n} (-1)^j y_j(x) \int_a^x \frac{\{\prod_{h<k,\, h\neq j\neq k} (m_k - m_h)\} y_0(u) \cdots y_{j-1}(u) y_{j+1}(u) \cdots y_n(u)}{\{\prod_{h<k} (m_k - m_h)\} y_0(u) \cdots y_n(u)} \cdot g(u)\, du$$

$$= (-1)^n \sum_{j=0}^{n} (-1)^j y_j(x) \int_a^x \frac{g(u)\, du}{\{\prod_{h=0}^{j-1} (m_j - m_h)\}\{\prod_{k=j+1}^{n} (m_k - m_j)\} y_j(u)}$$

$$= \sum_{j=0}^{n} \frac{e^{m_j x}}{\prod_{h=0,\, h\neq j}^{n} (m_j - m_h)} \int_a^x e^{-m_j u} g(u)\, du\ .$$

LINEAR DIFFERENTIAL EQUATIONS *(Continued from page 409.)*

It is seen from this and (2) that $y(x) = [m_0, \cdots, m_n; G_x]$ where

$$G_x(m) = e^{mx}\int_a^x e^{-mu}g(u)\,du = \int_a^x e^{m(x-u)}g(u)\,du = \int_0^{x-a} e^{mv}g(x-v)\,dv\ .$$

Thus $G_x = F_x$ with F_x given as in Remark 1, and it is seen from Remark 1 that Theorems 1 and 1′ hold for the case $m_h \neq m_k$ whenever $h \neq k$. The general result will be reduced to this special case by a continuity argument.

For $k = 0, 1, \cdots, D_0^k\xi_0 = D_x^k\xi_x|_{x=0}$ is a polynomial function of degree k (cf. the first paragraph of section 3). Hence

$$\left(\frac{d}{dx}\right)^k[m_0, \cdots, m_n; \xi_x]\Big|_{x=0} = [m_0, \cdots, m_n; D_0^k\xi_0] = 0$$

for all $m_0, \cdots, m_n \in C$ and $0 \leq k < n$ by Lemma 2. It follows from this and repeated application of (6′) (cf. (7)) that $\int_0^{x-a}[m_0, \cdots, m_n; \xi_u]g(x-u)\,du$ is $n+1$ times continuously differentiable in x (regardless of repeated m_j's) and that for each compact set $A \subset C$ the left member of the equation in Theorem 1 is continuous in $(m_0, \cdots, m_n)$ uniformly for $m_0, \cdots, m_n \in A$ (for x fixed), Theorem 1 must hold for all m_j's, mutually distinct or not.

8. Analogous theory for a complex domain. Let $M \subset C$ be open. In the previous theorems replace the continuous function $g : [a, b] \to C$ by the analytic function $g : M \to C$, and make other changes accordingly. The result is a parallel theory for a complex domain. The details will not be given here.

REFERENCES

1. E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.
2. K. S. Miller, *Engineering Mathematics*, Rinehart, New York, 1956.
3. L. M. Milne-Thomson, *The Calculus of Finite Differences*, Macmillan, London, 1933.
4. N.-E. Nörlund, *Leçons sur les Séries d'Interpolation*, Gauthiers-Villars, Paris, 1926.
5. T. Popoviciu, *Sur quelques propriétés des fonctions d'une ou de deux variables reelles*, Mathematica, vol. 8, 1934, pp. 1-85.

The University of Kansas

Edited by

Roy Dubisch

Articles intended for this department should be sent to *Roy Dubisch, Department of Mathematics, University of Washington, Seattle, Washington.*

A SIMPLE APPROACH TO THE FACTORIZATION OF INTEGERS

William Edward Christilles

In a previous issue of this magazine,[1] we demonstrated the usefulness of the binary quadratic form,

(1) $$x^2 + pqy^2 ,$$

in integers x and y with integral coefficients, in the factorization of integers of the form $4k+1$, where k is a positive integer. In this paper, which also will deal with the factorization of integers, we include a study of the binary quadratic form,

(2) $$px^2 + qy^2 ,$$

in integers x and y with integral coefficients, also of the form $4k+1$ where k is a positive integer. We further include the more general integral binary quadratic form,

(3) $$a\alpha^2 + b\alpha\gamma + c\gamma^2 ,$$

of discriminant D, where $D = 4ac - b^2 \neq -b^2$. By form (3) we mean that if two integers $N_1 = a_1\alpha_1^2 + b_1\alpha_1\gamma_1 + c_1\gamma_1^2$ and $N_2 = a_2\alpha_2^2 + b_2\alpha_2\gamma_2 + c_2\gamma_2^2$, in integers α_i and γ_i, where $i = 1$ or 2, with integral coefficients, have the same discriminant D, then they have the same form.

We begin by introducing the following three lemmas:

Lemma 1. If M is a positive integer of the form $4k+1$ and r_1 and r_2 are any pair of positive integral roots of M where $M = r_1r_2$, there exist integers u, m, v, n, p, and q such that:

(i) $$r_1 = pu^2 + qv^2$$

(ii) $$r_2 = m^2 + pqn^2 .$$

Proof: The proof is identical to the proof of Lemma 1 from our previous paper in the 1960 May-June issue of this magazine for $p = v = n = 1$.

Lemma 2. If M is a positive integer of the form $4k+1$ and r_1 and r_2 are any pair of positive integral roots where $M = r_1r_2$, such that:

(i) $$r_1 = pu^2 + qv^2$$

(ii) $$r_2 = m^2 + pqn^2 ,$$

then M has the form px^2+qy^2.

Proof:

$$M = r_1r_2 = (pu^2+qv^2)(m^2+pqn^2) = p(um+qvn)^2+q(unp-vm)^2 .$$

If we let $(um+qvn) = x$ and $(unp-vm) = y$, we have the form

$$M = px^2+qy^2 .$$

Lemma 3.[2] If a number is represented properly by a form $[a, b, c]$ of discriminant d, then any divisor of that number is represented by some form of the same discriminant d.

Note: For our purpose we let $-d = D$.

Now, before continuing to our theorems, we observe the following transformation:

Let

$$r_1 = a_1s_1^2+b_1s_1t_1+c_1t_1^2 = \frac{1}{4a_1}[(2a_1s_1+b_1t_1)^2+Dt_1^2] , \tag{4}$$

and

$$r_2 = a_2s_2^2+b_2s_2t_2+c_2t_2^2 = \frac{1}{4a_2}[(2a_2s_2+b_2t_2)^2+Dt_2^2] , \tag{5}$$

where $D = 4a_1c_1-b_1^2 = 4a_2c_2-b_2^2$. Thus,

$$4a_1r_1 = u^2+Dv^2 , \tag{6}$$

and

$$4a_2r_2 = m^2+Dn^2 , \tag{7}$$

where $u = (2a_1s_1+b_1t_1)$, $m = (2a_2s_2+b_2t_2)$, $v = t_1$, and $n = t_2$.

Consequently,

$$16hM = x^2+Dy^2 , \tag{8}$$

where $h = a_1a_2$, $M = r_1r_2$, $x = (um+Dvn)$, and $y = (un-vm)$. Thus, we have transformed M into form (1) by multiplying M by $16h$. (Note: $p = 1$ and q is replaced by the discriminant D.)

We now state and prove the following two theorems:

Theorem 1. If M is a positive integer of the form $4k+1$ and r_1 and r_2 are any pair of positive integral roots of M where $M = r_1r_2$, there exist integers x, y, p, q, w, and B, such that:

(i) $$M = px^2+qy^2$$

(9) (ii) $$y^2-4pB(qB-x) = w^2$$

moreover, there are two integers v and n where $B = vn$, such that:

$$M = r_1r_2 = [p(\frac{w+y}{2np})^2+qv^2][(\frac{w-y}{2v})^2+pqn^2] , \tag{10}$$

where $(\frac{w+y}{2np})$ and $(\frac{w-y}{2v})$ are integers.

Proof: It follows from Lemma 1 and Lemma 2 that:

$$M = r_1 r_2 = px^2 + qy^2 = p(um+qvn)^2 + q(unp-vm)^2 ,$$

where $x = (um+qvn)$ and $y = (unp-vm)$. Eliminating u from the last two relations and solving the result for m, we obtain:

$$m = \frac{-y \pm \sqrt{y^2 - 4v(pqvn^2 - pnx)}}{2v} . \tag{11}$$

Similarly, eliminating m and solving for u, we obtain:

$$u = \frac{+y \pm \sqrt{y^2 - 4np(qnv^2 - xv)}}{2np} . \tag{11$'$}$$

Consequently, in both the above cases, we have,

$$y^2 - 4pB(qB - x) = w^2$$

where $B = vn$; and, solving for x, we obtain:

$$x = \frac{w^2 - y^2 + 4pqB^2}{4pB}$$

$$\therefore\ M = px^2 + qy^2 = p\left[\frac{w^2 - y^2 + 4pqB^2}{4pB}\right]^2 + qy^2$$

$$= p\left[\frac{w^4 - 2(yw)^2 + y^4}{(4pB)^2}\right] + \left[\frac{8q(pwB)^2 + 8q(pyB)^2}{16(pB)^2}\right] + \left[\frac{p4^2p^2q^2B^4}{4^2p^2B^2}\right]$$

$$= p\left[\frac{(w^2-2wy+y^2)(w^2+2wy+y^2)}{(4pB)^2}\right] + q\left[\frac{(w^2-2wy+y^2)+(w^2+2wy+y^2)}{4}\right] + pq^2B^2$$

$$= p\left[\frac{(w-y)^2(w+y)^2}{(2np)^2(2v)^2}\right] + q\left[\frac{(w-y)^2 + (w+y)^2}{4}\right] + pq^2v^2n^2$$

$$= \left[p\left(\frac{w+y}{2np}\right)^2 + qv^2\right]\left[\left(\frac{w-y}{2v}\right)^2 + pqn^2\right] ,$$

where $(\frac{w+y}{2np}) = u$ and $(\frac{w-y}{2v}) = m$, from (11) and (11)$'$. This completes the proof of the theorem.

Now, by solving the relation $y^2 - 4pB(qB-x) = w^2$, for B, we obtain:

$$B = \frac{-4px \pm \sqrt{(4px)^2 + 4^2pq(y^2 - w^2)}}{-8pq} = \frac{-px \pm \sqrt{(px)^2 + pq(y^2 - w^2)}}{-2pq} .$$

Consequently:

$$(px)^2 + pq(y^2 - w^2) = (pz)^2$$

or:

$$px^2 + qy^2 = pz^2 + qw^2 .$$

Thus, we have shown that if a number M can be expressed in more than one distinct way in the form $px^2 + qy^2$, for a given p and q with $pq > 0$, then M is composite and its factors easily determined.

Theorem 2. If M is an integer, represented properly by form, (3) $a\alpha^2 + b\alpha\gamma + c\gamma^2$, of discriminant D, and r_1 and r_2 are any pair of positive integral roots of M where $M = r_1 r_2$, there exist integers x, y, h, w, D, and B, such that:

$$\text{(i)} \qquad 16hM = x^2 + Dy^2$$

$$\text{(12) (ii)} \qquad y^2 - 4B(DB - x) = w^2$$

moreover, there are integers v, n, a_1, and a_2, where $B = vn$ and $h = a_1 a_2$, such that

$$M = r_1 r_2 = \left[\frac{\left(\frac{w+y}{2n}\right)^2 + Dv^2}{4a_1}\right]\left[\frac{\left(\frac{w-y}{2v}\right)^2 + Dn^2}{4a_2}\right] \tag{13}$$

where $\left(\frac{w-y}{2v}\right)$, $\left(\frac{w+y}{2n}\right)$, $\left[\frac{\left(\frac{w+y}{2n}\right)^2 + Dv^2}{4a_1}\right]$ and $\left[\frac{\left(\frac{w-y}{2v}\right)^2 + Dn^2}{4a_2}\right]$ are integers.

Proof: It follows from Lemma 3 and the transformation expressed by (4), (5), (6), (7), and (8), that:

$$16hM = 16hr_1 r_2 = [4a_1(a_1 s_1^2 + b_1 s_1 t_1 + c_1 t_1^2)][4a_2(a_2 s_2^2 + b_2 s_2 t_2 + c_2 t_2^2)]$$

$$= (u^2 + Dv^2)(m^2 + Dn^2) = x^2 + Dy^2 ,$$

where $r_1 = a_1 s_1^2 + b_1 s_1 t_1 + c_1 t_1^2$, $r_2 = a_2 s_2^2 + b_2 s_2 t_2 + c_2 t_2^2$, $D = 4a_1 c_1 - b_1^2 = 4a_2 c_2 - b_2^2$, $u = (2a_1 s_1 + b_1 t_1)$, $m = (2a_2 s_2 + b_2 t_2)$, $v = t_1$, $n = t_2$, $h = a_1 a_2$, $M = r_1 r_2$, $x = (um + Dvn)$, and $y = (un - vm)$. Eliminating u, from the last two relations and solving the result for m, we obtain:

$$m = \frac{-y \pm \sqrt{y^2 + 4v(xn - Dvn^2)}}{2v} . \tag{14}$$

Similarly, eliminating m and solving for u, we get

$$u = \frac{+y \pm \sqrt{y^2 + 4n(xv - Dnv^2)}}{2n} . \tag{14$'$}$$

Consequently, in both cases, we have

$$y^2 - 4B(DB - x) = w^2 ,$$

where $B = vn$; and solving for x, we obtain:

$$x = \frac{w^2 + 4DB^2 - y^2}{4B}$$

$$\therefore\ 16\lambda M = x^2 + Dy^2 = \left[\frac{4DB^2 + w^2 - y^2}{4B}\right]^2 + Dy^2$$

$$= \left[\frac{w^4 - 2(wy)^2 + y^4}{(4B)^2}\right] + \left[\frac{8D(wB)^2 + 8D(yB)^2}{16B^2}\right] + \left[\frac{4^2D^2B^4}{4^2B^2}\right]$$

$$= \left[\frac{(w^2-2wy+y^2)(w^2+2wy+y^2)}{16v^2n^2}\right] + D\left[\frac{(w^2-2wy+y^2)+(w^2+2wy+y^2)}{4}\right] + D^2v^2n^2$$

$$= \left[\frac{(w-y)^2(w+y)^2}{16v^2n^2}\right] + D\left[\frac{(w-y)^2 + (w+y)^2}{4}\right] + D^2v^2n^2$$

$$= \left[\left(\frac{w+y}{2n}\right)^2 + Dv^2\right]\left[\left(\frac{w-y}{2v}\right)^2 + Dn^2\right],$$

where $\left(\frac{w-y}{2v}\right) = m$, and $\left(\frac{w+y}{2n}\right) = u$, from (14) and (14)′.

Thus,

$$16\lambda M = 16a_1a_2M = \left[\left(\frac{w+y}{2n}\right)^2 + Dv^2\right]\left[\left(\frac{w-y}{2v}\right)^2 + Dn^2\right]$$

or:

$$M = \frac{\left[\left(\frac{w+y}{2n}\right)^2 + Dv^2\right]\left[\left(\frac{w-y}{2v}\right)^2 + Dn^2\right]}{16a_1a_2}$$

$$= \left[\frac{\left(\frac{w+y}{2n}\right)^2 + Dv^2}{4a_1}\right]\left[\frac{\left(\frac{w-y}{2v}\right)^2 + Dn^2}{4a_2}\right],$$

from (6) and (7). This completes the proof of the theorem.

We conclude this paper with the following two examples:

Example 1. Let $M = 9{,}593$. We find that

$$M = px^2 + qy^2 = 3(\pm 54)^2 + 5(\pm 13)^2 .$$

Thus,

$$p = 3, \quad q = 5, \quad x = \pm 54, \quad \text{and} \quad y = \pm 13 .$$

Next, we attempt to satisfy (9) by successively substituting for B, in (9), the integers ± 1, ± 2, ± 3, $\cdots$. For $B = 2$, $p = 3$, and $q = 5$, we have:

$$y^2 - 4pB(qB - x) = (\pm 13)^2 - 4\cdot 3\cdot 2(5\cdot 2 - 54) = (\pm 35)^2 = (w)^2 .$$

Thus we find that for $p = 3$, $q = 5$, $v = 1$, $n = 2$, $y = -13$, and $w = 35$, we have:

$$M = \left[p\left(\frac{w-y}{2np}\right)^2 + qv^2\right]\left[\left(\frac{w+y}{2v}\right)^2 + pqn^2\right] = (53)(181) = 9{,}593 .$$

Example 2. Let $M = 11,221$. We find that M can be represented properly in form (3) as follows:

$$M = a\alpha^2 + b\alpha\gamma + c\gamma^2 = (11)^2 + (11)(100) + (100)^2 ,$$

where $D = 3$. Now we find by referring to table 1, on page 85 of *Introduction to the Theory of Numbers* by Leonard Eugene Dickson, that:

$$a_1 a_2 = 1 = h .$$

Thus,

$$16hM = 16M = 4[(\pm 122)^2 + 3(\pm 100)^2] = (\pm 244)^2 + 3(\pm 200)^2 ,$$

from (4), where $D = 3$, $x^2 = (\pm 244)^2$, and $y^2 = (\pm 200)^2$.

Then, by successively substituting, for B, in (12), the integers ± 1, ± 2, ± 3, $\cdots$, we find that for $B = -27$, we have:

$$y^2 - 4B(DB - x) = (\pm 200)^2 - 4(-27)[3(-27) - 244] = (\pm 70)^2 = (w)^2 .$$

$\therefore w = \pm 70$.

Thus we find for $y = 200$, $w = -70$, $D = 3$, $v = 27$, and $n = -1$, we have:

$$16M = \left[\left(\frac{200 - 70}{2(-1)}\right)^2 + 3(27)^2\right]\left[\left(\frac{200 + 70}{2(27)}\right)^2 + 3(-1)^2\right]$$

or

$$M = \left(\frac{6412}{4}\right)\left(\frac{28}{4}\right) = (1603)(7) = 11,221 .$$

REFERENCES

1. W. E. Christilles, A Note on the Factorization of Integers. Mathematics Magazine, Vol. 33, No. 5. May 1960, pp. 283-286.

2. Dickson, Leonard Eugene. *Introduction to the Theory of Numbers.* New York: Dover Publications, Inc., 1929, p. 95.

St. Mary's University
San Antonio, Texas

CURRENT PAPERS AND BOOKS

Edited by H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to *H. V. Craig, Department of Applied Mathematics, University of Texas, Austin, 12, Texas.*

ERRATUM:

In "A Numerical Congruence" on page 358 of the September-October 1961 issue, the factor $(h-k+1)$ should have been $(k+1)$. The congruence would then read

$$h(h-1)(h-2)\cdots(k+1) \equiv 1 \pmod{x} .$$

This might have been inferred from the six examples given.

BOOK REVIEWS

Recreational Mathematics Magazine. Bimonthly. Published and edited by Joseph S. Madachy. Box 1876, Idaho Falls, Idaho. 65¢ a copy, $3.00 per year (after March 1, 1962, $3.25 a year).

The first issue of this welcome addition to the field of mathematical periodicals appeared in February 1961. The fifth number (October 1961) is now off the press. Response in the forms of subscriptions, contributions and letters has been tremendous, clearly attesting to the need for the magazine and to its sustained quality. The business acumen of the founder and his personal enthusiasm for recreational mathematics augur well for the continued growth and long life of this worthy project.

Northrop's RIDDLES IN MATHEMATICS sparked Madachy's interest in the subject at an early age. The flame continued to burn with varying intensity during the acquisition of B.S. and M.S. degrees in chemistry at Western Reserve University, five years in the army, and short periods in high school and university teaching. It became a full-scale conflagration while he was doing chemical research into the problem of radioactive waste disposal. Following intensive investigation into publishing problems and after strong encouragement by Martin Gardner and others, J. S. Madachy decided to devote his full time to RMM. It was a fortunate decision for mathematics.

Recreational Mathematics Magazine is intended to fulfill the desire of many for a periodical entirely devoted to the lighter side of mathematics. Directed toward the mathematical hobbyist, many of its articles will hold the interest of the professional mathematician. The breadth of coverage may be judged from such representative titles as: Wager Problems, Mathematics of Music, The General Theory of Polyominoes, The 18 Perfect Numbers, Conics by Paper Folding, the Game is HOT, Pascal and Fibonacci,

Electronic Abstractions, Some Amazing Afghan Bands, A Look at the 4th Dimension, Circles and Spirals, The Haunted Checkerboards, Permutacrostics, and Magic Squares and Cubes. The familiar names C. Stanley Ogilvy, J. A. H. Hunter, Solomon W. Golomb, F. Emerson Andrews, Leo Moser, Mel Stover, Ali R. Amir-Moez, Brother Alfred, W. R. Ransom, Maxey Brooke, and Sidney Kravitz appear among the authors.

Departments dealing with Puzzles and Problems (and Answers), Alphametics, Word Games, Numbers Numbers Numbers, Book Reviews, Bibliographies, Readers' Research, and Letters to the Editor appear regularly. A page of Prime Numbers has appeared in each issue, the last prime printed being 25097.

The issues have averaged 60 well-arranged pages enclosed in an attractive cover. The typography is very good. The type is large enough to be read easily, and the pages are nicely broken up with well-drawn figures and cartoons. The mathematics is carefully checked by the editor and infrequent errors are promptly acknowledged.

When asked about the direction RMM would take, the editor replied, "Generally, in the direction pointed out by the readers. But, I will never let RMM leave the fields of puzzles and recreational mathematics. It must always remain a magazine to which almost anyone with reasonable intelligence could turn for some thought-provoking entertainment. I want RMM to be the source of the high school student's first appreciation for math – if he's never had it before. I want it to be the place where seriously inclined students – high school *and* college – may find, for the first time, that math is not just a deadly serious bit of business and that much of the best in mathematics arose from having fun, first."

With this philosophy, the editor of RMM deserves the active support of everyone who has an interest in mathematics – by subscribing, recommending and contributing. Every library, school and public, should have a complete file of RMM from issue No. 1 on. A subscription to RMM would be an inexpensive and welcome present to any person with an active and inquiring mind.

Charles W. Trigg

Automatic Data-Processing Systems. By Robert H. Gregory and Richard L. Van Horn. Wadsworth Publishing Co., xii + 705 pp., $8.95.

This book should be read in a somewhat backward manner, in that each chapter is followed by a careful summary of what that chapter covered. This reviewer suggests that you read the chapter summary before reading the chapter, since the book covers quite a wide variety of modern computer work. For example, Part 6 discusses the acquisition of equipment and its proper utilization, whereas Parts 4 and 5 discuss the principles of data-processing systems and system design. The orientation section, Part 1, pages 1 to 119, gives a general description of what happens in most data-processing systems. Part 2 discusses actual equipment, and

Part 3 the programming techniques and processing procedures. The last portion of the text proper, before the appendices, is devoted to a re-examination of what is going on and some comments on prospective developments. A very short section is devoted to mathematical models. The appendices include the history of computation and data-processing devices, a glossary of terms, and a series of questions and answers. These are in the nature of problems of the type which one might expect to find in any text book, but are clumped together at the back of the book where they can be used or not, depending upon the reader's inclination. The book is not apparently intended for the specialist in computer science, but rather for management in industry. It can well be read with profit by persons with little mathematical background.

Richard V. Andree

Partial Differential Equations and Continuum Mechanics. Edited by Rudolph E. Langer. The University of Wisconsin Press, Madison, 1961, xv + 397 pp., $5.00.

This volume consists of a transcript of a series of nineteen lectures delivered at an international conference conducted by the Mathematics Research Center, at the University of Wisconsin, from June 7 to 15, 1960. It also includes the abstracts of forty-five research papers presented at the conference principally by the investigators working in the United States.

Although the Mathematics Research Center at Madison is primarily oriented toward investigations immediately useful in scientific and technological applications, the majority of papers in this volume lie at the forefront of theoretical investigations in partial differential equations and mechanics of continua.

The titles of the nineteen lectures and their authors (in the order in which they appear in the book) are:

Aspects of Differential Equations in Mathematical Physics. Claus Müller, Technische Hochschule, Aachen, Germany.

The Angular Distribution of Eigenvalues of Non Self-adjoint Elliptic Boundary Value Problems of Higher Order. Shmuel Agmon, Hebrew University, Jerusalem, Israel.

Certain Indefinite Differential Eigenvalue Problems – The Asymptotic Distribution of their Eigenfunctions. Åke Pleijel, University of Lund, Sweden.

Bounds for Eigenvalues and the Method of Intermediate Problems. Alexander Weinstein, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland.

Linear Elliptic Equations of Higher Order in two Independent Variables and Singular Integral Equations, with Applications to Anisotropic Inhomogeneous Elasticity. Gaetano Fichera, Instituto Matematico, Università di Roma, Rome, Italy.

The Propagation of Surface Waves in Anisotropic Media. Robert

Stoneley, The University of Cambridge, England.

Finite Deformation of Plates into Shells. B. R. Seth, Indian Institute of Technology, Kharagpur, India.

Statistical Fluid Mechanics: Two-dimensional Linear Gravity Waves. J. Kampé de Fériet, The University of Lille, France.

Continuations of Laplace's Transformation; their Applications to Differential Equations. Jean Leray, College de France, Paris, France.

On the Regularity Problem for Elliptic and Parabolic Differential Equations. Jürgen Moser, New York University.

Atypical Partial Differential Equations. Hans Lewy, University of California, Berkeley.

Asymptotic Behavior of the Flow past a Body of a Compressible, Viscous or Electrically Conducting Fluid. Isao Imai, The University of Tokyo, Japan.

Transonic Gas Flow and the Equations of Mixed Type. Francesco G. Tricomi, Universitá di Torino, Turin, Italy.

Trans-sonic Nozzle Flows found by the Hodograph Method. T. M. Cherry, University of Melbourne, Australia.

On Existence of Solutions of Partial Differential Equations. Lars Hörmander, University of Stockholm, Sweden.

Existence and Differentiability Theorems for Variational Problems for Multiple Integrals. Charles B. Morrey, Jr., University of California, Berkeley.

Contribution to Mathematical Methods Applied in Fluid Mechanics. D. P. Riabouchinsky, Institut Superieur Technique Russe en France, Paris, France.

A Fuñctional Equation Related to the Boltzmann Equation and to the Equations of Gas Dynamics. J. M. Burgers, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland.

Parabolic Equations with Applications to Boundary Layer Theory. Karl Nickel, Institut für angewandte Mathematik, Karlsruhe, Germany.

The quality of the lectures and of some contributed papers is such that no serious scholar in the domain of partial differential equations and mechanics of continua should overlook this volume. The Mathematics Research Center at Madison is continuing to perform a useful service by organizing conferences and symposia in several vital fields of applied mathematics.

I. S. Sokolnikoff

Special Functions. By Earl D. Rainville. Macmillan, New York, 1960, vi + 365 pages, $11.75.

I would like to comment on H. S. Wall's review of Earl D. Rainville's new book *Special Functions*. (Math. Mag., Jan.-Feb., 1961)

It strikes me that the reviewer missed the central point of the book, that is to say the development of certain properties by means of a special

technique using generating functions, etc. It is definitely *not* the usual treatment, and I do not believe the book provides "approximately the 'bag of tools' which the usual engineering faculty believes desirable for their students to 'cover' in some course" as the reviewer says.

The treatment given the Bernoulli numbers, polynomials, and the things in that particular category is definitely sketchy, but I do not think this one part is the basis for an overall condemnation as the reviewer's article seems to imply. Also, in the treatment of the Laguerre polynomials, Hermite polynomials, Legendre polynomials, the reader will come on many things quite novel, not in the 'usual' treatments. Some of the exercises there are in quite a different category from the ones the reviewer quoted. What for example, is 'wrong' with the definitions used for Legendre polynomials? There is of course a lot of manipulative emphasis in the book, but it is apparent that it is introductory, not in the category of Whittaker and Watson, and gives an almost effortless approach to at least an introductory understanding of the definitely manipulative techniques which usually come not by rote work but by some kind of maturity which is very hard to come by these days.

I am not trying to write here a poetic defense of Rainville's book, because surely it is not without fault. But I disagree with the former reviewer. I would not like to see the readers of the Magazine misled into thinking that the book is just a standard old-fashioned cook-book of manipulative hoop-de-do. Rainville's book is definitely a good contribution to the literature.

H. W. Gould

BOOKS RECEIVED FOR REVIEW

Advanced Calculus. By John M. H. Olmstead. Appleton-Century-Crofts Inc., New York, 1961, xviii + 706 pp., $9.50.

Introduction to the La Place Transform. By Dio L. Holl, Clair G. Maple, and Bernard Vinograde. Appleton-Century-Crofts Inc., New York, 1959, viii + 174 pp.

Elementary Differential Equations. By William Ted Martin and Eric Reissner. Addison-Wesley Inc., Reading, Massa., 1961, xiii + 331 pp., $6.75.

Convex Figures. By I. M. Yaglom and V. G. Boltyanski (translated by Paul J. Kelly and Lewis F. Walton). Rinehart and Winston Inc., New York, 1961, xv + 301 pp., $4.75.

Problems of Continuum Mechanics (Contributions in Honor of N. I. Muskhelishvili). Society for Industrial and Applied Mathematics, Philadelphia, 1961, xx + 601 pp., $10.50.

Introduction to Matrices and Vectors. By Jacob T. Schwartz. McGraw-Hill Book Company, Inc., New York, 1961, x + 163 pp., $5.50.

A Modern Introduction to Logic. By L. S. Stebbing. Harper Brothers, Inc., 1961, xviii + 525 pp., $2.75 (paper bound).

Games of Strategy Theory and Applications. By Melvin Dresher. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1961, xii + 186 pp., $9.00.

Lattice Theory (Proceedings of Symposia in Pure Mathematics). Edited by R. P. Dilworth. American Mathematical Society, Providence, R. I., 1961, viii + 208 pp., $6.30.

Differential Geometry (Proceedings of Symposia in Pure Mathematics). Edited by C. B. Allendoerfer. American Mathematical Society, Providence, R. I., 1961, vii + 200 pp., $7.60.

Structure of Language and its Mathematical Aspects (Proceedings of the Symposia in Applied Mathematics). Edited by R. Jakobson. American Mathematical Society, Providence, R. I., 1961, vi + 279 pp., $7.80.

Nuclear Reactor Theory (Proceedings of the Symposia in Applied Mathematics). Edited by Garrett Birkhoff and Eugene P. Wigner. American Mathematical Society, Providence, R. I., 1961, v + 339 pp.

Measure, Lebesgue Integrals, and Hilbert Space. By A. N. Kolmogorov and S. V. Fomin. Academic Press, New York, 1961, xii + 147 pp., $4.00.

Linear Algebra and Group Theory. By Academician V. I. Smirnov (revised by Richard A. Silverman. McGraw Hill Book Company, Inc., New York, 1961, x + 464 pp., $12.50.

Numerical Weather Analysis and Prediction. By Philip D. Thompson. MacMillan Company, Inc., New York, 1961, xiv + 170 pp., $6.50.

Probability. By Frederick Mosteller, Robert E. K. Rourke, and George B. Thomas Jr. Addison-Wesley Publishing Company, Inc., Reading, Mass., 1961, xv + 319 pp., $5.00.

Probability with Statistical Applications. By Frederick Mosteller, Robert E. K. Rourke, and George B. Thomas Jr. Addison-Wesley Publishing Company, Inc., Reading, Mass., 1961, xv + 478 pp., $6.50.

The History of the Calculus and its Conceptual Development. By Carl B. Boyer. Dover Publications, Inc., New York, (reprint), 1949(1959), vi + 364 pp., $2.00 (paper bound).

A Short Account of the History of Mathematics. By W. W. Rouse Ball. Dover Publications, Inc., New York, (reprint), 1908(1960), xxiv + 522 pp., $2.00 (paper bound).

Fundamentals of Mathematics. By Thomas L. Wade and Howard E. Taylor. McGraw-Hill Book Company, Inc., New York, 1961, xiii + 428 pp., $6.75.

Arithmetic for College Students. By L. J. Adams. Holt, Rinehart and Winston, Inc., New York, 1961, ix + 262 pp., $3.75.

Understanding Basic Mathematics. By Leslie H. Miller. Holt, Rinehart, and Winston, Inc., New York, 1961, x + 499 pp., \$6.25.

Fundamentals of College Mathematics. By John C. Brixey and Richard V. Andree. Holt, Rinehart, and Winston, Inc., New York, xiv + 750 pp., \$8.95.

Evaluation in Mathematics. National Council of Teachers of Mathematics, Washington, 1961, iii + 428 pp., \$3.00.

College Algebra. By Paul K. Rees and Fred W. Sparks. McGraw-Hill Book Company, New York, 1961, xii + 428 pp., \$6.50.

Basic Analysis. By Stephen P. Hoffman. Holt, Rinehart, and Winston, Inc., New York, 1961, xi + 459 pp., \$6.50.

Simplified Calculus. By F. L. Westwater. The Macmillan Company, New York, 1960, xv + 160 pp., \$3.50.

Careers in Mathematics. National Council of Teachers of Mathematics, Washington, 1961, 28 pp., \$.25.

The Solution of Equations in Integers. By A. O. Gelfond. W. H. Freeman and Co., San Francisco, 1961, viii + 63 pp. \$1.00.

Some Ideas About Number Theory. By I. A. Barnett. National Council of Teachers of Mathematics, Washington, 1961, vi + 71 pp., \$1.40.

The Challenge of Science Education. Edited by Joseph S. Roucek. Philosophical Library, New York, 1959, 491 pp., \$10.00.

Tables of Coefficients for Obtaining the Second Derivative Without Differences. By H. E. Salzer and P. T. Roberson. Convair-Astronautics, San Diego, 1961, 25 pp.

Tables for Bivariate Osculatory Interpolation Over a Cartesian Grid. By H. E. Salzer and G. M. Kimbro. Convair-Astronautics, San Diego, 1961, 40 pp.

Tables of Osculatory Integration Coefficients. By H. E. Salzer, D. C. Shoultz, and E. P. Thompson. Convair-Astronautics, San Diego, 1960, 43 pp.

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India Ink and exactly the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

PROPOSALS

460. *Proposed by Robert P. Goldberg, Brooklyn, New York.*

Given a regular polynon $A_1A_2 \cdots A_n$ inscribed in a unit circle. Prove that

$$\prod_{i=2}^{n} A_1A_i = n\ .$$

461. *Proposed by Maxey Brooke, Sweeny, Texas.*

Prove the cryptarythm $[NA] \cdot [CL] \neq SALT$ in the base 6.

462. *Proposed by Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania.*

It is well known that $\max(x, y) = \frac{1}{2}\{|x-y| + x + y\}$. Find a similar expression for $\max(x, y, z)$.

463. *Proposed by L. Silverman, Fort Meade, Maryland.*

A debt exists between each pair of members of a certain club, though no member is in debt to all of the others. A member is considered a "deadbeat" if every other member is a creditor or a creditor of a creditor of his. Show that the club has at least three deadbeats.

464. *Proposed by D. Rameshwar Rao, Secunderabad, Andhra Pradesh, India.*

If $a^2 + b^2 = c^2$, prove that one can find integers $d, e, f, g, \cdots$ and $d_1, e_1, f_1, g_1, \cdots$ such that $a^2 + b^2 + d^2 = d_1^2$, $a^2 + b^2 + d^2 + e^2 = e_1^2$, $a^2 + b^2 + d^2 + e^2 + f^2 = f_1^2$ and so on, where $d, d_1; e, e_1; f, f_1; \cdots$ are consecutive integers.

465. *Proposed by Glenn D. James, Los Angeles City College.*

If the probability that it rains in a certain town on an April day is .2 and the probability that it rains there on an April day is .6 if it is known to have rained the day before; furthermore, the probability, if it did not rain the day before, that it rains is .05, what is the probability that it will rain on two consecutive April days?

466. *Proposed by E. P. Starke, Rutgers University.*

Let A be a positive integer of r digits given by

$$A = \sum_{i=1}^{r} x_i 10^{i-1}$$

and define

$$D(A) = \sum_{i=1}^{r} x_i^2 .$$

By $D^n(A)$, where n is a positive integer, we mean the result of applying the operator, D, n successive times to A. Prove that for every A there exists an n such that $D^n(A) = 1$ or 4.

(Cf. E 718 *American Mathematical Monthly.*)

SOLUTIONS

The Spider and the Fly

439. [March 1961] *Proposed by Maxey Brooke, Sweeney, Texas.*

A fly is resting on the floor on the north side of the base of a circular column. It suddenly realizes that a spider is resting diametrically opposite him. The fly starts crawling due north. At the same time the spider starts traveling due east. As everyone knows, a spider can crawl three times as fast as a fly.

When the fly has crawled 3 inches, he sees the spider just emerging from behind the curve of the column. He realizes that all is lost and freezes on the spot. The spider turns and crawls in a straight line to the fly and devours him. How far does the spider travel?

Solution by Herbert R. Leifer, Pittsburgh, Pennsylvania.

Let C and E be the starting positions of the spider and fly respectively, A the position from which the fly sees the spider at B. Since

$$\overline{OD}^2 = \overline{AO}^2 - \overline{DA}^2 , \quad \overline{OD} = \overline{OE} , \quad \overline{AO} = \overline{OE} + \overline{EA} , \quad \overline{EA} = 3 ,$$

we readily find

$$\overline{OD} = \frac{(\overline{DA}^2 - 9)}{6} .$$

Since

$$\Delta AOD \cong \Delta ABC , \quad \overline{OD} : \overline{CB} = \overline{DA} : \overline{AC} ;$$

also since

$$\overline{BD} = \overline{CB} = 3\overline{EA} = 9 , \quad \overline{AC} = 2\overline{OD} + 3 ,$$

we readily find

$$\overline{DA}^3 - 9\overline{DA} - 162 = 0 \quad \text{or} \quad (\overline{DA} - 6)(\overline{DA}^2 + 6\overline{DA} + 27) = 0 .$$

The spider travels the distance

$$\overline{CB} + \overline{BD} + \overline{DA} = 9 + 9 + 6 = 24 \text{ inches} .$$

Also solved by Brother Alfred, St. Mary's College, California; Merrill Barnebey, University of North Dakota; Donald K. Bissonnette, Florida State University; Robert H. Clark, Naval Underwater Ordnance Station, Newport, Rhode Island; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; Monte Dernham, San Francisco, California; M. S. Klamkin, AVCO, Wilmington, Massachusetts; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; Alan Sutcliffe, Knottingley, Yorkshire, England; J. B. Thomas, Albion, Pennsylvania; Paul D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C.; C. W. Trigg, Los Angeles City College; Dale Woods, Oklahoma State University; and the proposer.

Circle Packing

440. [March 1961] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Consider a packing of circles of radius r such that each is tangent to its six surrounding circles. Let a larger circle of radius R be drawn concentric with one of the small circles. If N is the number of small circles contained in the larger circle, prove that

$$N = 1 + 6n + 6 \sum_{p=1}^{n} [\tfrac{1}{2}(\sqrt{4n^2 - 3p^2} - p)]$$

where $n = [\tfrac{1}{2}(\frac{R}{r} - 1)]$, the square brackets designating the greatest integer function.

Solution by Alan Sutcliffe, Knottingley, Yorkshire, England.

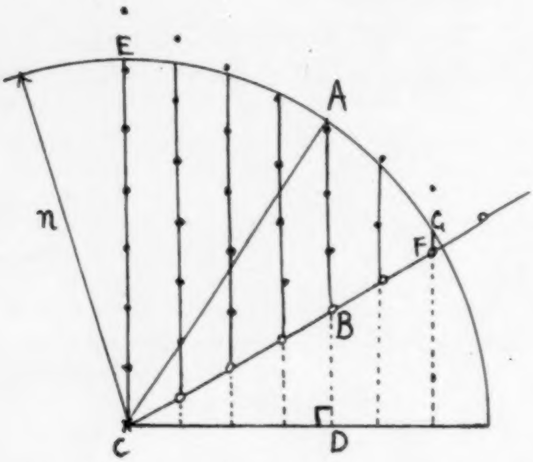
The expression is not quite correct. For example when $\frac{R}{r} = 2\sqrt{3} + 1$ we have $n = 1$ and hence $N = 7$, while the correct value is $N = 13$. The correct expression is

$$N = 1 + 6[n] + 6 \sum_{p=1}^{[n]} [\tfrac{1}{2}(\sqrt{4n^2 - 3p^2} - p)] = 1 + 6 \sum_{p=0}^{[n]} [\sqrt{n^2 - (3/5)p^2} - \tfrac{p}{2}] ,$$

where $n = \tfrac{1}{2}(\frac{R}{r} - 1)$.

To prove this we shall first assume unit distance between adjacent centers, and find the number of centers within a circle of radius r. Because of the triangular nature of the array of centers, we need consider only one of the six similar sectors of the circle as shown in the diagram, where the centers marked $\circ$ are in the adjoining sector and the common center C

is in no sector. Clearly the number of centers contained within the sector



is the sum of the integral part of the lengths, such as AB, from CE to FG. Let $CB = p$, which will be an integer. Then, since angle $BCD = 30°$, $CD = (\sqrt{3}/2)p$ and $BD = p/2$. As $AC^2 = AD^2 + CD^2$ we have

$$n^2 = (AB + \frac{p}{2})^2 + \frac{3}{4}p^2 .$$

Hence

$$AB = \sqrt{n^2 - (3/4)p^2} - \frac{p}{2} .$$

The number of centers within the sector is the sum of the integral part of this from $p = 0$ to $[n]$. Since there are six sectors and the common center C, we have

$$N = 1 + 6\sum_{p=0}^{[n]} [\sqrt{n^2 - (3/4)p^2} - \frac{p}{2}] .$$

Now in fact the centers are not unit distance, but $2r$ apart. So that a radius $R = 2rn$ will contain N centers. Thus a radius $R = 2rn + r$ will contain N circles, giving $n = \frac{1}{2}(\frac{R}{r} - 1)$, which completes the proof.

Chasles' Theorem Generalized

441. [March 1961] *Proposed by Vladimir F. Ivanoff, San Carlos, California.*

A skew quadrilateral $ABCD$ lies entirely on a ruled quadric surface. Show that for any point $P(x, y, z)$ on the quadric,

$$\frac{PA \cdot PC \sin\beta \sin\delta}{PB \cdot PD \sin\alpha \sin\gamma} = \lambda$$

where α, β, γ, and δ are the dihedral angles whose edges are PA, PB, PC, and PD respectively and λ is a constant.

Solution by the proposer.

Let the coordinates of the points A, B, C, and D be $(x_1 y_1 z_1)$, $(x_2 y_2 z_2)$, $(x_3 y_3 z_3)$ and $(x_4 y_4 z_4)$ respectively. Then the equation of the quadric surface may be thrown into the following form:

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = \lambda \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

where λ is a constant (value of which may be determined if we are given an additional point $(x_0 y_0 z_0)$ of the quadric). Each of the four determinants given here represents six times the volume of a particular tetrahedron. Thus, the first determinant is equal to six times the volume of the tetrahedron $PABC$. We can give another expression to the volume of this tetrahedron, namely:

$$\begin{aligned} 6V &= (\text{Area } PAB) \times (\text{altitude from } C \text{ to } AB\text{-plane}) \\ &= \overline{PA} \cdot \overline{PB} \sin \measuredangle APB \cdot \overline{PC} \cdot \sin \measuredangle BPC \cdot \sin \beta \\ &= \overline{PA} \cdot \overline{PB} \cdot \overline{PC} \sin \measuredangle APB \cdot \sin \measuredangle BPC \cdot \sin \beta\,. \end{aligned}$$

Similarly, the next three determinants have the values as follows:

$$\overline{PA} \cdot \overline{PC} \cdot \overline{PD} \cdot \sin \measuredangle APD \cdot \sin \measuredangle DPC \cdot \sin \delta$$

$$\overline{PA} \cdot \overline{PB} \cdot \overline{PD} \cdot \sin \measuredangle DPA \cdot \sin \measuredangle APB \cdot \sin \alpha$$

$$\overline{PB} \cdot \overline{PC} \cdot \overline{PD} \cdot \sin \measuredangle BPC \cdot \sin \measuredangle CPD \cdot \sin \gamma$$

respectively. The substitution of these values into the equation of the quadric after cancellations, results in the formula in question.

Unbalanced Forces

442. [March 1961] *Proposed by C. W. Trigg, Los Angeles City College.*

A 10-gram mass and a 120-gram mass are connected by a strong, light 100 cm. thread. They are placed on the horizontal top of a table 100 cm. high, with the 10-gram mass just over an edge to which the taut string is then perpendicular. The system is then released. Where will the heavier mass strike the floor? (The coefficient of friction with the table top is 0.04.)

Solution by J. B. Thomas, Albion, Pennsylvania.

First compute the velocity of the 120 gram mass the instant it leaves the table top. The unbalanced forces acting on the system are the force of gravity on the 10 gram mass and the force of friction on the 120 gram mass. Since $\Sigma F = ma$, we have

$$10\,\text{gm} \times 980\,\text{cm/sec}^2 - 0.04 \times 120\,\text{gm} \times 980\,\text{cm/sec}^2 = 130\,\text{gm} \times a$$

which gives $a = 39.2\,\text{cm/sec}^2$. The velocity is given by

$$v = \sqrt{2as} = \sqrt{2\times 39.2\times 100}\ \text{cm/sec} = 88.5\ \text{cm/sec}\ .$$

Since the time of flight from the edge of the table is given by

$$t = \sqrt{2h/a} = \sqrt{2\times 100/980}\ \text{sec} = .452\ \text{sec}\ ,$$

the 120 gram mass will strike the floor at a horizontal distance

$$vt = 88.5\times .452 = 40.0\ \text{cm}$$

from the table top.

Also solved by Robert H. Clark, Naval Underwater Ordnance Station, Newport, Rhode Island; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; Merle E. Riley, Marietta College, Ohio; and the proposer.

Characteristic Roots

443. [March 1961] *Proposed by B. Weesakul, The University of Western Australia.*

Show that the characteristic roots of the $n\times n$ matrix

$$\begin{vmatrix} 0 & p & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 \\ 0 & q & 0 & p & \cdots & 0 \\ 0 & & & & & \\ \vdots & & & q & 0 & p \\ 0 & 0 & \cdots & 0 & q & 0 \end{vmatrix}$$

are $S_v = 2(pq)^{1/2}\cos\frac{v\pi}{n+1}$, $p>0$, $q>0$ and $v = 1, 2, \cdots, n$.

Solution by Douglas H. Moore, California State Polytechnic College. The required roots are the roots, in λ, of the determinant:

$$D_n = \begin{vmatrix} -\lambda & p & 0 & 0 & \cdots & 0 \\ q & -\lambda & p & 0 & \cdots & 0 \\ 0 & q & -\lambda & p & \cdots & 0 \\ \vdots & & & & & \\ 0 & \cdots & & q & -\lambda & p \\ 0 & \cdots & & 0 & q & -\lambda \end{vmatrix} .$$

Expanding in terms of the elements of the first row:

$$D_n = -\lambda D_{n-1} - pqD_{n-2}\ , \quad n > 3\ .$$

By defining $D_0 = 1$, we may consider the recursion system:

(1) $$D_{n+2} + \lambda D_{n+1} + pqD_n = 0\ , \quad D_0 = 1\ , \quad D_1 = -\lambda\ .$$

As a check on the appropriateness of this definition of D_0, set $n = 0$ in (1) to obtain: $D_2 + \lambda D_1 + pqD_0 = 0$ or: $D_2 = \lambda^2 - pq$, which agrees with the

above definition of D_2. The solution of (1) is:

$$D_n = (-\sqrt{pq})^n \frac{\sin (n+1)\omega}{\sin \omega}, \tag{2}$$

where

$$\lambda = 2\sqrt{pq} \cos \omega \tag{3}$$

as may be found by the classical method of evaluating arbitrary constants in the appropriate complementary function. From (2) it is seen that the roots of D_n are those values of λ which make $\sin(n+1)\omega$ vanish, i. e. which make:

$$(n+1)\omega = \pi,\ 2\pi,\ 3\pi,\ \cdots,\ n\pi$$

where 0 and $(n+1)\pi$ are excluded to avoid a zero denominator in (2). Then:

$$\omega = \frac{\pi}{n+1},\quad \frac{2\pi}{n+1},\quad \cdots,\quad \frac{n\pi}{n+1}.$$

Therefore, from (3):

$$\lambda_\nu = 2\sqrt{pq} \cos \frac{\nu\pi}{n+1}, \quad \nu = 1, 2, \cdots, n.$$

Also solved by Cheong Seng Hoo, Auckland, New Zealand; M. S. Klamkin, AVCO, Wilmington, Massachusetts; Paul D. Thomas, U.S. Coast and Geodetic Survey, Washington, D. C.; and the proposer.

Binomial Coefficients

444. [March 1961] *Proposed by Melvin Hochster, New York.*

If k and n are integers such that $0 \leq k \leq n$, $[p]$ represents the greatest integer part of p, and $\binom{q}{-1} = 0$, prove that

$$\sum_{i=0}^{k} \left\{\binom{k}{[\frac{1}{2}k]-i} - \binom{k}{[\frac{1}{2}k]-i-1}\right\} \left\{\binom{2n-k}{[n-\frac{1}{2}k]-i} - \binom{2n-k}{[n-\frac{1}{2}k]-i-1}\right\} = \binom{2n}{n} - \binom{2n}{n-1}.$$

Solution by L. Carlitz, Duke University.

1. Let $k = 2r$. Then

$$\sum_{j=0}^{r} \left\{\binom{2r}{r-j} - \binom{2r}{r-j-1}\right\} \left\{\binom{2n-2r}{n-r-j} - \binom{2n-2r}{n-r-j-1}\right\}$$

$$= \sum_{j=0}^{r} \left\{\binom{2r}{r+j}\binom{2n-2r}{n-r-j} + \binom{2r}{r-j-1}\binom{2n-2r}{n-r+j+1}\right\} - \sum_{j=0}^{r} \left\{\binom{2r}{r+j+1}\binom{2n-2r}{n-r-j} + \binom{2r}{r-j}\binom{2n-2r}{n-r+j+1}\right\}$$

$$= \sum_{j=0}^{2r}\binom{2r}{j}\binom{2n-2r}{n-j} - \sum_{j=0}^{2r}\binom{2r}{j}\binom{2n-2r}{n-j+1} = \binom{2n}{n} - \binom{2n}{n+1} = \binom{2n}{n} - \binom{2n}{n-1}.$$

2. Let $k = 2r+1$. Then

$$\sum_{j=0}^{r}\left\{\binom{2r+1}{r-j}-\binom{2r+1}{r-j-1}\right\}\left\{\binom{2n-2r-1}{n-r-j-1}-\binom{2n-2r-1}{n-r-j-2}\right\}$$

$$= \sum_{j=0}^{r}\left\{\binom{2r+1}{r+j+1}\binom{2n-2r-1}{n-r-j-1}+\binom{2r+1}{r-j-1}\binom{2n-2r-1}{n-r+j+1}\right\}$$

$$-\sum_{j=0}^{r}\left\{\binom{2r+1}{r+j+1}\binom{2n-2r-1}{n-r-j-2}+\binom{2r+1}{r-j-1}\binom{2n-2r-1}{n-r+j}\right\}$$

$$= \sum_{j=0}^{2r+1}\left\{\binom{2r+1}{j}\binom{2n-2r-1}{n-j}-\binom{2r+1}{r}\binom{2n-2r-1}{n-r}\right\}$$

$$-\sum_{j=0}^{2r+1}\left\{\binom{2r+1}{j}\binom{2n-2r-1}{n-j-1}+\binom{2r+1}{r}\binom{2n-2r-1}{n-r-1}\right\}$$

$$= \binom{2n}{n}-\binom{2n}{n-1}\ .$$

Also solved by the proposer.

A Polar Relation

445. [March 1961] *Proposed by the late Victor Thebault, Tennie, Sarthe, France.*

Given in a plane a circle (O) on which one takes two points A and B, and a point M of which the power with respect to (O) is k. Prove that the polar of M with respect to (O) meets AB in a point P such that

$$\frac{PA}{PB} = \frac{\overline{MA}^2-k}{\overline{MB}^2-k}\ .$$

Solution by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.

Let the orthogonal projections of A and B on the line OPM be A', B'. Considering the directed segments we first have

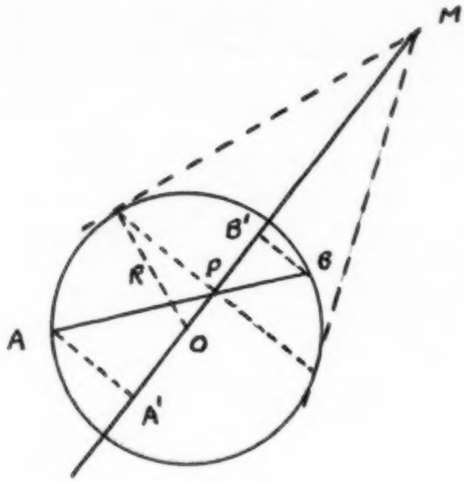
$$k = \overline{MO}\cdot\overline{MP}\ ,\quad (R^2 = \overline{OP}\cdot\overline{OM})$$

which is valid for all positions of M with regard to the circle (O). It is positive, zero or negative according as M is outside, on or inside (O).

Now,

$$\overline{MA}^2-k = \overline{AA'}^2+\overline{A'M}^2-(\overline{OM}^2-R^2) = (R^2-\overline{OA'}^2)+(\overline{A'O}+\overline{OM})^2-(\overline{OM}^2-R^2)$$
$$= R^2-\overline{OA'}^2+\overline{A'O}^2+2\overline{A'O}\cdot\overline{OM}+\overline{OM}^2-\overline{OM}^2+R^2 = 2(R^2+\overline{A'O}\cdot\overline{OM})$$

$$\overline{MA}^2 - k = 2(\overline{OP}\cdot\overline{OM} + \overline{A'O}\cdot\overline{OM}) = 2\overline{OM}(\overline{A'O}+\overline{OP}) = 2\overline{OM}\cdot\overline{A'P}\ .$$



Similarly

$$\overline{MB}^2 - k = 2\overline{OM}\cdot\overline{B'P}\ .$$

Dividing we have the required result

$$\frac{\overline{MA}^2 - k}{\overline{MB}^2 - k} = \frac{2\overline{OM}\cdot\overline{A'P}}{2\overline{OM}\cdot\overline{B'P}} = \frac{\overline{A'P}}{\overline{B'P}} = \frac{\overline{PA}}{\overline{PB}}\ .$$

Also solved by J. Gallego-Diaz, Universidad del 'Zulia, Maracaibo, Venezuela; James W. Mellender, University of Wisconsin; and Paul D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C.

Comment on Problem 425

425. [November 1960 and May 1961] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

If $n-1$ and $n+1$ are twin prime numbers, prove that $3\phi(n) \leqq n$ where ϕ denotes Euler's ϕ-function.

Comment by David A. Klarner, Napa, California.

The solution given by Dermott A. Breault contains an error. In the proof we find the statement, "If $n+1$ and $n-1$ are prime, n is even and a multiple of 3, so that for some m, $n = 6m$, and we have

$$\phi(n) = \phi(6)\phi(m) = 2\phi(m)\ ."$$

This is only true when $(6, m) = 1$. In fact, the twin primes 11, 13 yield

$$\phi(12) = \phi(6)\cdot\phi(2) = 2\ ,$$

but $\phi(12) = 4$. Therefore the method of proof given would have to be altered to make it valid.

Comment on Q 277

Q 277. [March 1961] *Comment by Charles T. Salkind, Polytechnic Institute of Brooklyn.* If we form the differences of the second, third, etc., orders for the given Fibonacci sequence we obtain, 1, 0, 1, 1, 2, 3, 5, ···, −1, 1, 0, 1, 1, 2, ···, and so forth. Certainly these do not correspond to the original sequence. If we exercise the privilege of removing the first term of the sequence of first differences and repeat this "tampering" with each successive sequence of differences, then the Fibonacci sequence does quality under the stated given conditions. If, however, some special "tampering" is permitted, then we can use a geometric progression with any positive r greater than 1 and are not limited to $r = 2$. The "tampering" in this case is to multiply the sequences of successive differences by $(1/(r-1))$. For example,

$$\text{sequence}: a,\ 3a,\ 9a,\ 27a,\ 81a,\ \cdots$$

$$\frac{1}{r-1}[\Delta_j] = \frac{1}{2}(2a,\ 6a,\ 18a,\ \cdots) = a,\ 3a,\ 9a,\ \cdots$$

and

$$\frac{1}{r-1}[\Delta_j^n] = a,\ 3a,\ 9a,\ \cdots .$$

In fact the case given, where $r = 2$, is merely the case where $(1/(r-1))$ reduces to 1. It is possible to eliminate the restriction on integers given by Brother Alfred, since, for example, the sequence

$$\cdots,\ \frac{a}{4},\ \frac{a}{2},\ a,\ 2a,\ 4a,\ \cdots$$

yields sequences of differences of all orders which repeat the original sequence. If the sequence a, $2a$, $4a$, $8a$, ··· is replaced by another in which the terms are the sums of successive terms of the given sequence, two at a time, three at a time, etc., then the new sequence will also have the property described in the quicky. Finally, if the differencing, instead of being done between successive terms, is performed between, say the kth term and the first, the $(k+1)$st term and the second, and so on, then the required sequence is a, $a2^{1/k}$, $a2^{2/k}$, ··· or with the "tampering" mentioned above, a, $ar^{1/k}$, $ar^{2/k}$, ··· for any positive r greater than 1.

QUICKIES

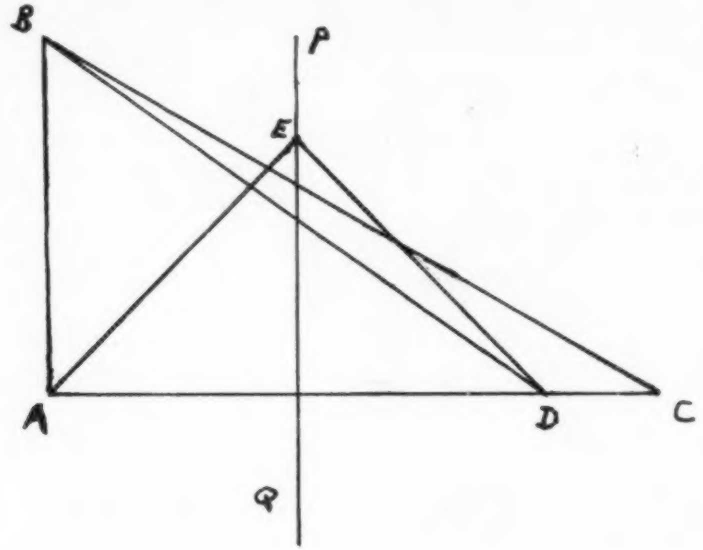
From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 288. Factor

$$x^8 - x^7y + x^6y^2 - x^5y^3 + x^4y^4 - x^3y^5 + x^2y^6 - xy^7 + y^8$$

[*Submitted by Anice Seybold.*]

Q 289. In the accompanying figure, AB is perpendicular to AC and half the length of BC; $BC = AC$; PQ is the perpendicular bisector of AD; $AE = AB$. Show that AED is a right triangle.



[*Submitted by Monte Dernham.*]

TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase or idea rather than upon a mathematical routine. Send us your favorite trickies.

T 46. The most common proof of the theorem, "The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides" involves drawing a parallel line and using similar triangles. Devise a shorter proof using areas. [*Submitted by Robert P. Goldberg.*]

T 47. Show that in any polygon there exist two sides whose ratio lies between 1/2 and 2. [*Submitted by M. S. Klamkin.*]

T 48. Write a formula $y = f(x)$ such that for all positive integral values of x, the corresponding value of y is a prime number. [*Submitted by Frank Hawthorne.*]

(Answers to Quickies and Solutions for Trickies are on page 436.)

ANSWERS *to Quickies on page 434.*

A 288. We have

$$x^9+y^9 = (x+y)(x^8-x^7y+x^6y^2+\cdots+y^8)$$
$$= (x^3+y^3)(x^6-x^3y^3+y^6)$$
$$= (x+y)(x^2-xy+y^2)(x^6-x^3y^3+y^6)\ .$$

Hence

$$x^8-x^7y+x^6y^2+\cdots+y^8 = (x^2-xy+y^2)(x^6-x^3y^3+y^6)\ .$$

A 289. If $AB = 1$, $AC = \sqrt{3} = BD$. Hence $AD = \sqrt{2}$. But $DE = AE = AB = 1$. Thus AED is a right angle.

SOLUTIONS *for Trickies on page 435.*

S 46. Since the given triangle is divided into two triangles which have equal altitudes we have

$$\frac{k_1}{k_2} = \frac{m}{n}\ .$$

But

$$k_1 = \frac{1}{2}ad\sin\alpha \quad \text{and} \quad k_2 = \frac{1}{2}bd\sin\alpha\ ,$$

$$\frac{k_1}{k_2} = \frac{m}{n} = \frac{\frac{1}{2}ad\sin\alpha}{\frac{1}{2}bd\sin\alpha} = \frac{a}{b}\ .$$

S 47. Assume that it is not true. Then the largest side would be greater than the sum of all the other sides. That is

$$ar^n > a+ar+ar^2+\cdots+ar^{n-1} \quad \text{if} \quad r \geqq 2\ .$$

S 48. Any of the following will do.

$$y = 3(\sin^2 x+\cos^2 x)$$
$$y = 4+(-1)^x$$
$$y = 5+2\sin\frac{\pi-x}{2}\ .$$

PROFESSIONAL OPPORTUNITIES

IN MATHEMATICS

Fifth Edition *September 1961*

A completely revised version of an article which appeared originally in the AMERICAN MATHEMATICAL MONTHLY.

32 pages, paper covers

25¢ for single copies; 20¢ each for orders of five or more.

Send orders to :

HARRY M. GEHMAN, *Executive Director*
Mathematical Association of America
University of Buffalo
Buffalo 14, New York

The Mathematical Association of America

The Association is a national organization of persons interested in mathematics at the college level. It was organized at Columbus, Ohio, in December 1915 with 1045 individual charter members and was incorporated in the State of Illinois on September 8, 1920. Its present membership is over 11,000, including more than 500 members residing in foreign countries.

Any person interested in the field of mathematics is eligible for election to membership. Annual dues of $5.00 includes a subscription to the American Mathematical Monthly. Members are also entitled to reduced rates for purchases of the Carus Mathematical Monographs and for subscriptions to several journals.

Further information about the Association, its publications and its activities may be obtained by writing to:

HARRY M. GEHMAN, *Executive Director*
MATHEMATICAL ASSOCIATION OF AMERICA
University of Buffalo
Buffalo 14, New York